LTCC course on Potential Theory, Spring 2012

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Contents (five two-hour lectures):

The four lectures follow closely a textbook on Potential Theory in the Complex Plane by T. Ransford, apart from material on harmonic measure which has been borrowed from a lecture course Introduction to Potential Theory with Applications, by C. Kuehn. The material in lecture 5 is borrowed from a survey Logarithmic Potential Theory with Applications to Approximation Theory by E. B. Saff (E-print: arXiv:1010.3760).

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1 Harmonic Functions

(Lecture 1, 20 Feb 2012)

1.1 Harmonic and holomorphic functions

Let D be domain (connected open set) in $\mathbb{C}$. We shall consider complex-valued and real valued functions of $z = x + iy \in D$. We shall write $h(z)$ or often simply h to denote both a function of complex z and a function in two variables x and y.

Notation: h_x will be used to denote the partial derivative of h with respect to variable x, similarly h_{xx}, h_{yy}, h_{xy} and h_{yx} are the second order partial derivatives of f.

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Here come our main definition of the day. Recall that $\mathbb{C}^2(D)$ denotes the space of functions on D with continuous second order derivatives.

Definition A function $h : D \to \mathbb{R}$ is called harmonic on D if $h \in \mathbb{C}^2(D)$ and $h_{xx} + h_{yy} = 0$ on D.

Examples

(a) $h(z) = |z|^2 = x^2 + y^2$ is not harmonic anywhere on $\mathbb{C}$ as $h_{xx} + h_{yy}$=2.

(b) $h(z) = \ln(|z|^2)$ is harmonic on $D = \mathbb{C}\backslash\{0\}$. Indeed,

$$\ln|z| = \ln\sqrt{x^2+y^2} \quad \text{and} \quad (h_{xx}+h_{yy})(z) = 0 \quad \text{if} \quad z \neq 0.$$

(c) $h(z) = \operatorname{Re} z = x^2 - y^2$ is harmonic on $\mathbb{C}$.

Recall that a complex-valued function f is called *holomorphic* on a domain D if it is complex differentiable in a neighbourhood of any point in D. The existence of a complex derivative is a very strong condition: holomorphic functions are actually infinitely differentiable and are represented by their own Taylor series. For a function $f(z) = h(z) + ik(z)$ to be homomorphic the following conditions are necessary and sufficient

$$\text{Cauchy-Riemann equations:} \quad h_x = k_y \quad \text{and} \quad h_y = -k_x$$

Thm 1. *Let D be a domain in $\mathbb{C}$. If f is holomorphic on D then $h = \operatorname{Re} f$ is harmonic on D.*

Proof. This follows from the Cauchy-Riemann equations. □

The converse is also true but only for simply connected domains D, i.e. when every path between two arbitrary points in D can be continuously transformed, staying within D, into every other with the two endpoints staying put.

Thm 2. *If h is harmonic on D and D is simply-connected then $h = \operatorname{Re} f$ for some holomorphic function f on D. This function is unique up to an additive constant.*

Proof. We first settle the issue of uniqueness. If $h = \operatorname{Re} f$ for some holomorphic f, say $f = h + ik.$, then by the Cauchy-Riemann,

$$f' = h_x + ik_x = h_y - ih_y \ . \tag{1.1}$$

Hence, if such f exists, then it is completely determined by the first order derivatives of h, and, therefore, is unique up to an additive constant.

Equation (1.1) also suggest a way to construct the desired f. Define $g = h_y - ih_y$. Then $g \in C(D)$ and g satisfies the Cauchy-Riemann equations because $h_{xx} = -h_{yy}$ (h is harmonic) and $h_{xy} = h_{yx}$. Therefore g is holomorphic in D. Now fix $z_0 \in D$, and define f to be the anti-derivative of g:

$$f(z) = h(z_0) + \int_{z_0}^{z} g(z)dz \ ,$$

with the integral being along a path in D connecting z and z_0. As D is simply connected, Cauchy's theorem asserts that the integral does not depend on the choice of path. By construction, f is holomorphic and

$$f' = g = h_x - ih_y \ .$$

Writing $\tilde{h} = \operatorname{Re} f$, by the Cauchy-Riemann for f,

$$f' = \tilde{h}_x - i\tilde{h}_y.$$

On comparing the two equations, one concludes that $\tilde{h}_x = h_x$ and $\tilde{h}_y = h_y$. Therefore $h - \tilde{h}$ is a constant. Since h and $\tilde{h}$ are equal at z_0, they are equal throughout. □

One corollary of these theorems is that every harmonic function is differentiable infinitely many times.

Corollary 3. *If h is a harmonic function on a domain D, then $h \in C^\infty(D)$.*

Another important corollary of Thm 2 is a property of harmonic functions that will be later used to define the important class of subharmonic functions.

Let $\Delta(w, r)$ denote the disk of radius r about w, and $\bar{\Delta}(w, r)$ denote its closure,

$$\Delta(w, r) = \{z : |w - z| < r\} \ , \quad \bar{\Delta}(w, r) = \{z : |w - z| \leq r\} \ .$$

The boundary $\partial\Delta$ of $\Delta(w, r)$ is the set

$$\partial\Delta(w, r) = \{z : \quad z = w + re^{i\theta}, \, \theta \in [0, 2\pi)\} \ .$$

Thm 4. *(Mean-Value Property) Let h be a function harmonic in an open neighbourhood of the disk $\bar{\Delta}(w, r)$. Then*

$$h(w) = \frac{1}{2\pi} \int_0^{2\pi} h(w + re^{i\theta})d\theta \ .$$

Proof. Function h is harmonic on $\Delta(w, \rho)$ for some $\rho > r$. By Thm 2 there exists a function f which is holomorphic on $\Delta(w, \rho)$ and such that $h = \operatorname{Re} f$. By Cauchy's integral formula for f,

$$f(w) = \frac{1}{2\pi i} \oint_{|z-w|=r} \frac{f(z)}{z-w}\, dz \ .$$

Introducing the parametrisation $z = w + re^{i\theta}$ for the integration path above (θ runs from 0 to 2π), with $dz = ire^{i\theta} d\theta$, one arrives at

$$f(w) = \frac{1}{2\pi} \int_0^{2\pi} f(w + re^{i\theta}) d\theta \ .$$

The result now follows on taking the real parts of both sides. □

Recall that every holomorphic function is completely determined on a domain by its values in a neighbourhood of a single point. A similar property holds for harmonic functions.

Thm 5. *(Identity Principle) Let h be a harmonic function on a domain D in $\mathbb{C}$. If $h = 0$ on a non-empty open subset U of D then $h = 0$ throughout D.*

Proof. Set $g = h_x - ih_y$. Then as in the proof of Thm 2, g is holomorphic in D. Since $h = 0$ on U then so is g. Hence, by the Identity Principle for the holomorphic functions $g = 0$ on D, and consequently, $h_x = h_y = 0$ on D. Therefore h is constant on D, and as it is zero on U, it must be zero on D. □

Note that for the holomorphic functions a stronger identity principle holds: if f is holomorphic on D and vanishes on D at infinite number of points which have a limiting point in D, then f vanishes on D throughout. This is not the case for harmonic functions, e.g. the harmonic function $h = \operatorname{Re} z$ vanishes on the imaginary axis, and only there.

The theorem below asserts that harmonic functions do not have local maxima or minima on open sets U unless they are constant. Moreover, if a harmonic function is negative (positive) on the boundary of an open U it will be negative (positive) throughout U.

Thm 6. *(Maximum Principle) Let h be a harmonic function on a domain D in $\mathbb{C}$.*

(a) If h attains a local maximum in D then h is constant.

(b) Suppose that D is bounded and h extends continuously to the boundary ∂D of D. If $h \leq 0$ on ∂D then $h \leq 0$ throughout D.

Proof. Suppose that h attains a local maximum in D. Then there exists a open disk $\tilde{\Delta}$ such that $h \leq M$ in $\tilde{\Delta}$ for some M. Consider the set $K = \{z \in \tilde{\Delta} : h(z) = M\}$. Since h is continuous, K is closed in Δ, as the set of points where a continuous function takes a given value is closed. It is also non-empty. Suppose there exists a boundary point ζ of K in $\tilde{\Delta}$. As K is closed, $h(\zeta) = M$, and, as $\tilde{\Delta}$ is open, one can find a disk $\Delta(\zeta, \rho)$ of radius ρ about ζ such that $\Delta(\zeta, \rho) \subset \tilde{\Delta}$ and such that there exist points on the circumference of $\Delta(\zeta, \rho)$ where $h < M$ (for if not, then ζ would not be at the boundary of K). Let $\zeta + \rho e^{i\theta_0}$ be one such point. Since the complement of K is an open set, there exists a neighbourhood of $\zeta + \rho e^{i\theta_0}$ where $h < M$. Therefore one can find $\epsilon > 0$ and $\delta > 0$ such that for all θ such that $|\theta - \theta_0| < \delta$:

$$h(\zeta + \rho e^{i\theta}) < M - \epsilon \ . \tag{1.2}$$

Now, write the mean-value property for h at point ζ,

$$\begin{aligned} h(\zeta) &= \frac{1}{2\pi} \int_0^{2\pi} h(\zeta + \rho e^{i\theta}) d\theta \\ &= \frac{1}{2\pi} \int_{|\theta - \theta_0| < \delta} h(\zeta + \rho e^{i\theta}) d\theta + \frac{1}{2\pi} \int_{|\theta - \theta_0| \geq \delta} h(\zeta + \rho e^{i\theta}) d\theta \end{aligned}$$

The first integral above is $\leq 2\delta(M - \epsilon)$ in view of (1.2) and the second $\leq (2\pi - 2\delta)M$ because $h \leq M$ throughout $\tilde{\Delta}$. Therefore one concludes that

$$h(\zeta) < \frac{1}{2\pi} [2\delta(M - \epsilon) + (2\pi - 2\delta)M] < M \ ,$$

which is a contradiction as by the construction $h(\zeta) = M$. Hence $K = \{z \in \tilde{\Delta} : h(z) = M\}$ cannot have boundary points in $\tilde{\Delta}$ and therefore $K = \tilde{\Delta}$. Hence h is constant on $\tilde{\Delta}$. Then, by the identity principle Thm 5, h is constant on D, and Part (a) is proved.

To prove Part (b), observe that h is continuous on $\bar{D}$ by assumption. Since $\bar{D}$ is compact, then h must attain a (global) maximum at some point $w \in \bar{D}$. If $w \in \partial D$ then $h(w) \leq 0$ by assumption, and so $h \leq 0$ on D. If $w \in D$ then, by Part (a), h is constant on D. Hence, by continuation, h is constant on $\bar{D}$ of which ∂D is a subset, so once again $h \leq 0$. □

It follows from the identity principle that if two harmonic functions coincide in a neighbourhood of a single point of a domain D then they coincide everywhere in D. We shall now establish a stronger property. If two harmonic functions coincide on the boundary of a disk about a point in D then they coincide throughout D.

We first settle the issue of uniqueness in the case of the entire D. The corresponding statement is known as the Uniqueness Theorem

Thm 7. *(Uniqueness Theorem) Let D be a bounded domain in $\mathbb{C}$ and h_1 and h_2 are two harmonic functions that extend continuously to the boundary ∂D of D. If $h_1 = h_2$ on ∂D then these two functions are equal throughout D.*

Proof. Consider the function $h = h_1 - h_2$. This function is harmonic and, by construction, $h = 0$ on ∂D. Therefore, by the Maximum Principle (Thm 6) $h \leq 0$ on D. Applying now the Maximum Principle to the function $-h$ we conclude that $h = 0$ on D, hence $h_1 = h_2$ there. $\square$

By evoking the Identity Principle (Thm 6) one can make a stronger statement about uniqueness.

Corollary 8. *Let D be a domain in $\mathbb{C}$ and $z \in D$. If h_1 and h_2 are two functions harmonic on D and such that $h_1 = h_2$ on the boundary of a disk about z in D then these two functions are equal throughout D.*

1.2 Poisson integral

Since the boundary values of harmonic function determine this function uniquely (under the assumption of continuity on the boundary), it is natural to ask the question about reconstructing the harmonic function from its boundary values. A slightly more general problem is known as the Dirichlet Problem.

Definition Let D be a domain in $\mathbb{C}$ and let $\phi : \partial D \to \mathbb{R}$ be a continuous function . The Dirichlet problem is to find a function h harmonic on D and such that $\lim_{z \to \zeta} h(z) = \phi(\zeta)$ for all $\zeta \in D$.

The uniqueness of solution of the Dirichlet Problem follows is asserted by Thm 7. The question of existence is more delicate. We shall solve here one important case of a circular domain (disk) when the positive answer comes via an explicit construction. It uses the so-called Poisson Kernel.

Now we shall set about finding the harmonic function on a disk from its values on the disk boundary.

Definition The following real-valued function of two complex variables z and ζ

$$P(z,\zeta) = \mathrm{Re}\left(\frac{\zeta + z}{\zeta - z}\right) = \frac{1-|z|^2}{|\zeta - z|^2} \quad (|z| < 1, |\zeta| = 1)$$

is known as the Poisson kernel.

Lemma 9. *(Properties of the Poisson kernel)*

(a) $P(z,\zeta) > 0$ *for* $|z| < 1$ *and* $|\zeta| = 1$*;*

(b) $\frac{1}{2\pi}\int_0^{2\pi} P(z, e^{i\theta})d\theta = 1$*;*

(c) $\sup_{|\zeta-\zeta_0|\geq\delta} P(z,\zeta) \to 0$ *as* $z \to \zeta_0 \quad (|\zeta_0| = 1$ *and* $\delta > 0)$ *.*

Proof. Parts (a) and (c) are obvious (for part (c) observe that as z approaches ζ_0, the top of the fraction defining the Poisson kernel goes to zero and the bottom is bounded away from zero by the triangle inequality $|z-\zeta| \geq |\zeta - \zeta_0| - |z - \zeta_0|$).

A calculation is required for Part (b) . Observe that

$$\frac{1}{2\pi}\int_0^{2\pi} P(z, e^{i\theta})d\theta = \mathrm{Re}\left(\frac{1}{2\pi i}\int_{|\zeta|=1} \frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta}\right).$$

Since

$$\frac{\zeta + z}{(\zeta - z)\zeta} = \frac{2}{\zeta - z} - \frac{1}{\zeta},$$

Part (b) now follows from Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

for holomorphic functions. □

The Poisson kernel is defined for z in the unit disk about the origin. By making an affine change of variables $u = \frac{z-w}{\rho}$ one can map a disk about w of radius ρ in z-plane to the unit disk about the origin in u-plane. This justifies the following definition.

Definition Let $\phi : \Delta(w,\rho) \to \mathbb{R}$ be continuous function. Then its Poisson Integral $P_\Delta\phi$ is defined by

$$P_\Delta\phi(z) = \frac{1}{2\pi}\int_0^{2\pi} P\left(\frac{z-w}{\rho}, e^{i\theta}\right)\phi(w + \rho e^{i\theta})d\theta \quad (z \in \Delta)$$

In the polar coordinates for z in the disk about w, this takes the form

$$P_\Delta \phi(w + re^{it}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r \cos(\theta - t) + r^2} \phi(w + \rho e^{i\theta}) d\theta$$

Lemma 10. *(Properties of the Poisson Integral)*

(a) $P_\Delta \phi$ *is harmonic on* Δ *(this statement also holds true for Lebesgue integrable* ϕ*).*

(b) $\lim_{z \to \zeta_0} P_\Delta \phi(z) = \phi(\zeta_0)$ *for every* $\zeta_0 \in \partial\Delta$ *(for this statement to hold true the continuity of* ϕ *at* ζ_0 *is essential).*

Proof. It follows from the definition of the Poisson kernel that $P_\Delta \phi$ is the real part of a holomorphic function of z, hence it is harmonic.

To simplify notations, we shall prove Part (b) for $w = 0$ and $\rho = 1$, i.e. in the case when Δ is the unit disk about the origin[2]. In which case

$$P_\Delta \phi(z) = \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{i\theta}) \phi(e^{i\theta}) d\theta \ .$$

By making use of properties (a) and (b) of the Poisson kernel

$$|P_\Delta \phi(z) - \phi(\zeta_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} P(z, e^{i\theta}) |\phi(e^{i\theta}) - \phi(\zeta_0)| d\theta \ .$$

Now, we split the integral above in two. One over the range of values of θ for which $|e^{i\theta} - \zeta_0| < \delta$ and the other over the complementary range. For any $\epsilon > 0$ the former can be smaller than $\epsilon > 0$ by the choice of δ. This is because ϕ is continuous. And for fixed δ the latter tends to zero as z approaches ζ_0 in view of property (c) of the Poisson Integral. Since ϵ is arbitrary, the limit in Part (b) follows. □

As an immediate consequence of Lemma 10 we obtain a formula that allows to recover values of a harmonic function in a disk from its values on the disk boundary. This result, which is an analogue of the Cauchy integral formula for holomorphic functions, is fundamental.

Thm 11. *(Poisson Integral Formula) If* h *is harmonic in a neighbourhood of the disk* $\bar{\Delta}(w, \rho)$*, then for* $r < \rho$*, i.e. inside the disk*

$$h(w + re^{it}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r \cos(\theta - t) + r^2} h(w + \rho e^{i\theta}) d\theta$$

[2]This will suffice as we can always change variables.

Proof. Since h is harmonic in a neighbourhood of $\bar{\Delta}(w, \rho)$, it is continuous on the boundary $\partial\Delta$. Denote its restriction to $\partial\Delta$ by ϕ, i.e., $\phi = h|_{\partial\Delta}$. Then, by Lemma 10, $P_\Delta\phi(z)$ is harmonic on Δ, extends continuously to the boundary of Δ and coincides there with h. Therefore, by the Uniqueness Theorem 8 $h = P_\Delta\phi$. □

Writing the Poisson Integral Formula in the centre of the disk, i.e. for $r = 0$, one recovers the Mean-Value Property of harmonic functions which we established earlier. Thus, the Poisson Integral Formula can be viewed as a generalisation of the Mean-Value Property.

1.3 Positive harmonic functions

The Poisson integral formula allows to obtain useful inequalities for positive harmonic functions. Note that if a non-negative harmonic function attains a minimum value zero on a domain, it is zero throughout the domain. So the class of non-negative harmonic functions on a domain consists of all positive functions and a zero function.

Thm 12. *(Harnack's Inequality) Let h be a positive harmonic function on the disk $\Delta(w, \rho)$ of radius ρ about w. Then for $r < \rho$ and all t*

$$\frac{\rho - r}{\rho + r} h(w) \leq h(w + re^{it}) \leq \frac{\rho + r}{\rho - r} h(w)$$

i.e. the values of h in the disk $\Delta(w, \rho)$ are bounded below and above by multiples of the value of h at the centre w of the disk, with both bounds only depending on the distance to the centre.

Proof. Choose any s such that $r < s < \rho$ and apply the Poisson integral formula to to h on $\Delta(w, s)$. As h is positive, we have

$$\begin{aligned} h(w + re^{it}) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{s^2 - r^2}{s^2 - 2sr\cos(\theta - t) + r^2} h(w + se^{i\theta}) d\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{s^2 - r^2}{(s - r)^2} h(w + se^{i\theta}) d\theta \,. \end{aligned}$$

Simplifying the fraction and applying the mean-value property one obtains the desired upper bound. The lower bound is proved similarly. □

Recall that any function that is holomorphic on $\mathbb{C}$ and bounded in absolute value is a constant. As an immediate corollary of Harnack's inequalities one obtains an analogue of this result for harmonic functions.

Corollary 13. *(Liouville Theorem) Every harmonic function on $\mathbb{C}$ that is bounded below or above is constant.*

Proof. It will suffice to show that every positive harmonic function on $\mathbb{C}$ is constant. Let h be positive and harmonic on $\mathbb{C}$. By Harnack's inequalities applied for $\Delta(0,\rho)$,

$$h(z) \leq \frac{\rho + |z|}{\rho - |z|}\, h(0)\ .$$

Letting $\rho \to \infty$ and keeping z fixed, one concludes that $h(z) \leq h(0)$ for any $z \in \mathbb{C}$, hence h attains a maximum at 0. The Maximum Principle (Thm 7) then implies that h must be constant. □

Harnack's inequality on disks extends to general domains and can be used to define a distance between two points.

Corollary 14. *Let D be a domain in $\mathbb{C}$ and $z, w \in \mathbb{C}$. Then there exists a number τ such that, for every positive harmonic function on D,*

$$\tau^{-1} h(w) \leq h(z) \leq \tau\, h(w)\ . \tag{1.3}$$

Proof. Given z, w we shall write $z \sim w$ if there exists τ such that (1.3) holds for all positive functions harmonic on D. It is apparent that $\sim$is an equivalence relation. That is, (i) $z \sim z$; (ii) if $z \sim w$ then $w \sim z$; and (iii) if $z \sim w$ and $w \sim u$ then $z \sim u$. Consider the corresponding equivalence classes. By the Harnack's inequality they are open sets. As D is connected, there can only be one such equivalence class, hence (1.3) holds for all z, w. □

Definition (Harnack Distance) Let D be a domain in $\mathbb{C}$. Given $z, w \in D$, the Harnack distance between z and w is the smallest number $\tau_D(z, w)$ such that for every positive harmonic function h on D

$$\tau_D(z, w)^{-1} h(w) \leq h(z) \leq \tau_D(z, w)\, h(w)\ .$$

The Harnack distance is an useful notion and can be used to deduce an important theorem about convergence of positive harmonic functions, below. However, we do not have time to discuss it in detail. Instead we just state two of its properties.

Lemma 15. *(Properties of the Harnack Distance)*

(a) If $D_1 \subset D_2$ then $\tau_{D_2}(z, w) \leq \tau_{D_1}(z, w)$ for all $z, w \in D_1$.

(b) If D is a subdomain of $\mathbb{C}$ then $\ln \tau_D(z, w)$ is a continuous semi-metric on D (continuity meaning that $\ln \tau_D(z, w) \to 0$ as $z \to w$).

We shall finish this section with several important results about convergence of a sequence of harmonic functions. Proofs are not given here and can be found in Ransford's book.

Firstly, local uniform convergence of harmonic functions implies that the limiting function is also harmonic.

Thm 16. *If $(h_n)_{n\geq 1}$ is a sequence of harmonic functions on a domain D converging locally uniformly to a function h, then h is also harmonic on D.*

Non-increasing sequences of harmonic functions always converge (locally uniformly), a result known as Harnack's theorem.

Thm 17. *(Harnack's Theorem) Let $(h_n)_{n\geq 1}$ be non-decreasing sequence of harmonic functions on a domain D, i.e., $h_1 \leq h_2 \leq h_3 \leq \ldots$. Then either $h_n \to \infty$ locally uniformly, or else $h_n \to h$ locally uniformly, where h is harmonic.*

For positive sequences of harmonic functions one can only guarantee convergence of a subsequence.

Thm 18. *Let $(h_n)_{n\geq 1}$ be a sequence of positive harmonic functions on a domain D. Then either $h_n \to \infty$ locally uniformly, or else some subsequence $h_{n_j} \to h$ locally uniformly, where h is harmonic.*

Exercises 1

(1.1a) Show that the Poisson kernel is given by

$$P(re^{it}, e^{i\theta}) = \sum_{n=-\infty}^{+\infty} r^{|n|} e^{in(t-\theta)} \quad (r < 1, 0 \leq t, \theta < 2\pi) \; .$$

Use this to derive an alternative proof if Lemma 9 (b).

(1.1b) Show that it $\phi : \partial\Delta(0,1) \to \mathbb{R}$ is an integrable function, then

$$P_\Delta \phi(re^{it}) = \sum_{n=-\infty}^{+\infty} a_n r^{|n|} e^{int} \quad (r < 1, 0 \leq t < 2\pi) \; ,$$

where $(a_n)_n$ is a bounded sequence of complex numbers.

(1.1c) Assume now that ϕ is continuous. Writing $\phi_r(e^{it}) = P_\Delta\phi(re^{it})$, show that $\phi_r \to \phi$ uniformly on the unit circle as $r \to 1$, and deduce that $\phi(e^{it})$ can be uniformly approximated by trigonometric polynomials $\sum_{n=-N}^{N} b_b e^{int}$.

1.2 (Harnack distance) Let $\Delta = \Delta(w, 1)$. Show that

$$\tau_\Delta = \frac{1 + |z - w|}{1 - |z - w|} \quad (z \in \Delta) \; .$$

Hint: Use Harnack's inequality to establish an upper bound and then show that this bound is attained on the positive harmonic function $h(z) = P(z - w, \zeta)$.

2 Subharmonic Functions

(Lecture 2, 27 Feb 2012)

There are two standard approaches to define subharmonic functions. One is to require that $u_{xx} + u_{yy} \geq 0$ in the sense of distribution theory, matching the characteristic property of the harmonic functions. And the other one is via a submean property, matching the mean value property of harmonic functions. We shall follow the second approach.

Recall that the harmonic functions are continuous. However, subharmonic functions are not required to be continuous. For if they were that would be too restrictive as continuity is not preserved when taking limits of functions with $u_{xx} + u_{yy} \geq 0$. Obviously, one need to require some kind of regularity to have a meaningful theory, and it appears that semicontinuity suffices.

2.1 Upper semicontinuous functions

Recall that a function f on a metric space X is called continuous at x if for any given positive ϵ one can find a open ball $\Delta(x, \delta)$ of radius δ about x, $\Delta(x, \delta) = \{y : \text{dist}(x, y,) < \delta\}$, such that

$$f(x) - \epsilon \leq f(y) \leq f(x) + \epsilon \quad (y \in \Delta(x, \delta)) \ .$$

The inequality above is two-sided. If only one-sided inequality holds then the function is said to be semi-continuous.

Definition (Upper semi-continuity) A function $f : X \to [-\infty, +\infty)$ is called *upper semi-continuous* at $x \in X$ if for any given positive ϵ one can find a positive δ such that

$$f(y) \leq f(x) + \epsilon \quad (y \in \Delta(x, \delta)) \ .$$

Note that upper semi-continuous functions are allowed to take value $-\infty$. This is consistent with the definition above.

A function f is said to be upper semi-continuous on X if it has this property at every $x \in X$. An equivalent definition for f to be upper semi-continuous on X is to require that

$$\limsup_{y \to x} f(y) \leq f(x) \quad (x \in X) \ . \tag{2.1}$$

Yet another equivalent definition is to require the sets $\{x \in D : f(x) < \alpha\}$ be open in D for every $\alpha \in \mathbb{R}$.

Obviously, every continuous function is also upper semi-continuous. Below are a few examples of functions that are upper semi-continuous but not continuous.

Examples

(a) $f(z) = \ln|z|$, $z \in \mathbb{C}$. f is not continuous at $z = 0$.

(b) $f(x) = |x|$ if $x \neq 0$ and $f(0) = 1$ (x real or complex).

(c) $f(x) = \sin(1/x)$ if $x \neq 0$ and $f(0) = 1$.

(d) The characteristic function of a closed set in D.

Obviously, if f and g are upper semi-continuous functions, so is their sum $f + g$ and $\max(f(z), g(z))$.

Thm 19. *Let f be upper semi-continuous. Then f is bounded above on compact sets and attains its upper bound in every compact set.*

Proof. Based on the Bolzano-Weierstrass theorem and is same as for continuous functions. Let $M = \sup_{x\in K} f(x)$, where M may be $+\infty$. By the definition of sup, there exists a sequence (x_n) such that $f(x_n) \to M$ as $n \to \infty$. If K is compact then (x_n) contains a subsequence converging to a point $x \in K$. It follows from (2.1) that $M \leq f(x)$, hence M is finite. Also since $f(x) \leq M$ on K, one concludes that $f(x) = M$. □

Thm 20. *(Monotone Approximation by Continuous Majorants) If f is upper semi-continuous and bounded above on X then sequence of continuous functions (f_n such that $f_1 \geq f_2 \geq f_3 \geq \ldots \geq f$ and $f = \lim_{n\to\infty} f_n$ on X. The convergence is pointwise.*

Proof. In the singular case when $f = -\infty$ we can simply take $f_n = -n$. Suppose now that $f \not\equiv -\infty$. Define

$$f_n(x) = \sup_{y\in D} \left(f(y) - n\,\mathrm{dist}(y, x)\right) .$$

Obviously $f_n \geq f_{n+1}$ and $f_n \geq f$ for every n. These functions also satisfy the inequality

$$f_n(x) \leq f_n(w) + n\,\mathrm{dist}(x, w) \quad (x, w \in X) , \tag{2.2}$$

from which it follows that f_n is continuous on X for every n. For, interchanging x and w one gets $|f(x) - f(w)| \leq \mathrm{dist}(x, w)$.

To prove (2.2), let us fix w. By definition of f_n, for every $\epsilon > 0$ there exists y_ϵ such that

$$f_n(w) - \epsilon \leq f(y_\epsilon) - n \operatorname{dist}(y_\epsilon, w) \leq f_n(w) \ . \tag{2.3}$$

Now,

$$\begin{aligned} f_n(x) &\geq f(y_\epsilon) - n \operatorname{dist}(y_\epsilon, x) \\ &= f(y_\epsilon) - n \operatorname{dist}(y_\epsilon, w) + n \operatorname{dist}(y_\epsilon, w) - n \operatorname{dist}(y_\epsilon, x) \\ &\geq f_n(w) - \epsilon - \operatorname{dist}(x, w) \ , \end{aligned}$$

where on the last step we have used (2.3) and the triangle inequality $\operatorname{dist}(y_\epsilon, w) \leq \operatorname{dist}(y_\epsilon, x) + \operatorname{dist}(x, w)$. Therefore for every $\epsilon > 0$,

$$f_n(w) \leq f_n(x) + \operatorname{dist}(x, w) + \epsilon \ .$$

By letting $\epsilon \to 0$, one obtains (2.2). So f_n are continuous.

It remains to prove that f_n converge to f. Fix x. By definition of f_n, for every n, we can find x_n such

$$f_n(x) - \frac{1}{n} \leq f(x_n) - n \operatorname{dist}(x_n, x) \ . \tag{2.4}$$

After rearranging for $\operatorname{dist}(x_n, x)$:,

$$n \operatorname{dist}(x_n, x) \leq \frac{1}{n} - f_n(x) + f(x_n) \leq 1 - f(x) + \sup_{x \in X} f(x) \ .$$

As f is bounded above, it follows that $x_n \to x$ as $n \to \infty$. Letting $n \to \infty$ in (2.4), and making use of (2.1),

$$\lim_{n \to \infty} f_n(x) \leq \limsup_{n \to \infty} f(x_n) \leq f(x) \ .$$

On the other hand, $\lim_{n \to \infty} f_n(x) \geq f(x)$ because $f_n(x) \geq f(x)$ for every n. Hence $\lim_{n \to \infty} f_n(x) \geq f(x)$, and theorem follows. □

2.2 Subharmonic functions and their properties

Definition (Subharmonic Functions). Let U be an open subset of $\mathbb{C}$. A function u is called *subharmonic* is u is upper semi-continuous and satisfies the local submean inequality. Namely, for any $w \in U$ there exists $\rho > 0$ such that

$$u(w) \leq \frac{1}{2\pi} \int_0^{2\pi} u(w + \rho e^{it}) dt \quad (0 \leq r < \rho) \ . \tag{2.5}$$

Note, that according to this definition $u \equiv -\infty$ is a subharmonic function.

The integral in (2.5) is understood as the Lebesgue integral and is well defined for upper semi-continuous functions. Indeed, $\int u = \int u^+ - \int u^-$, where $u^\pm = \max(\pm u, 0)$. By Thm 19 , u^+ is bounded. Thus if $\int u^-$ is finite then the whole integral is finite too. If $\int u^- = +\infty$ then the whole integral $\int u = -\infty$. We shall see later (Corollary 28) that the latter can only happen only if $u \equiv -\infty$.

The following theorem provides an important example of subharmonic function.

Thm 21. *If f is holomorphic on D then* $\ln|f|$ *is subharmonic on D.*

Proof. $\ln|f|$ is upper semi-continuous, so one only needs to verify the local sub-mean property. Consider a point $w \in D$. If $f(w) \neq 0$, then $\ln|f|$ is harmonic near w and hence (2.5) follows from the mean value property of harmonic functions. If $f(w) = 0$ then $\ln|f(w)| = -\infty$ and (2.5) is satisfied anyway. □

Note that if u and v are subharmonic then so is $\alpha u + \beta v$ for all $\alpha, \beta \geq 0$, as well as $\max(u, v)$.

The following result is central and is an extension of the corresponding property of the harmonic functions to subharmonic functions (however, spot the differences ...)

Thm 22. *(Maximum Principle) Let u be subharmonic on a domain D in $\mathbb{C}$.*

(a) If u attains a global maximum in D then u is constant.

(b) If D is bounded and $\limsup_{z\to\zeta} u(z) \leq 0$ for all $\zeta \in \partial D$, then $u \leq 0$ on D.

Proof. Suppose that u attains a maximum value of M on D. Define

$$A = \{z \in D : u(z) < M\}, \quad K = \{z \in D : u(z) = M\} .$$

The set A open because u is upper semi-continuous. The set K is open too because of the local submean property for subharmonic functions (any sufficiently small circle about $z \in K$ must lie in K, for if not then there is a circle that intersects with A, and since A is open the intersection will contain a segment of finite length hence the mean value integral will be $< 2\pi M$ in violation of the local submean property). By assumption, A and K partition D. Since D is connected, one of the two sets must be empty. The set K is non-empty by assumption, therefore $A = \emptyset$, and Part (a) is proved.

To prove part (b), let us extend u to the boundary of D by $u(\zeta) = \limsup_{z\to\zeta} u(z)$, for $\zeta \in \partial D$. Then u is upper semi-continuous on $\bar{D}$. Since $\bar{D}$ is compact by Thm 19) u attains a maximum on $\bar{D}$. If the maximum point is in D, then $u = 0$ on $\bar{D}$ by Part (a). If the maximum point is at the boundary of D then $u \leq 0$ on $\bar{D}$. □

Comparison to the maximum principle for harmonic functions:

- local max for harmonic, global for subharmonic
- min or max for harmonic functions, only max for subharmonic

The subharmonic function $u = \max(\operatorname{Re} z, 0)$ attains a local maximum and a global minimum but is not constant.

The following theorem explains the name for subharmonic functions.

Thm 23. *(Harmonic Majoration) Let D be a bounded domain in $\mathbb{C}$ and suppose that u is subharmonic in D and h is harmonic there. Then*

$$\text{`}\limsup_{z\to\zeta}(u-h)(z) \leq 0 \text{ for all } \zeta \in \partial D' \quad \Longrightarrow \quad \text{`}u \leq h \text{ on } D\text{'} . \tag{2.6}$$

Proof. The function $u - h$ is subharmonic, hence the result follows from the maximum principle for subharmonic functions, Part (b) of Thm 22. □

Recall that harmonic functions can be represented via the Poisson integral formula. Correspondingly, subharmonic functions are bounded from above by the Poisson integral (you might expected this coming ...)

Thm 24. *(Poisson Integral Inequality) If u is subharmonic in a neighbourhood of $\bar{\Delta}(w, \rho)$ then for $r < \rho$*

$$u(w + re^{it}) \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r\cos(\theta - t) + r^2}\, u(w + \rho e^{i\theta})d\theta . \tag{2.7}$$

Proof. By Thm 20, there are continuous functions $\phi_n : \partial\Delta \to \mathbb{R}$ such that $\phi_n \downarrow u$ on $\partial\Delta$. Since ϕ_n is continuous, the function $P_\Delta\phi_n$ is harmonic and $\lim_{z\to\zeta}(P_\Delta\phi_n)(z) \to \phi_n(\zeta)$ for $\zeta \in \partial D$. Hence, (recall that $\phi_n \geq u$ for every n)

$$\limsup_{z\to\zeta}(u - P_\Delta\phi_n) \leq u - \phi_n \leq 0 .$$

Therefore, by the harmonic majoration theorem, Thm 23, $u \leq P_\Delta\phi_n$ on Δ. Letting $n \to \infty$ and using the monotone convergence theorem for Lebesgue integrals gives the desired inequality. □

As an immediate consequence of the Poisson integral inequality (put $r = 0$ in (2.7), one concludes that subharmonic functions satisfy the global submean inequality.

Corollary 25. *(Global Submean Inequality) If u is subharmonic on an open set U and $\bar{\Delta}(w, \rho) \in U$ then*

$$u(w) \leq \frac{1}{2\pi} \int_0^{2\pi} u(w + \rho e^{it}) dt \ .$$

Observe that (2.7) implies the local submean inequality (2.5), so we have a sequence of implications

$$(2.5) \implies (2.6) \text{ for all harmonic } h \implies (2.7) \implies (2.5),$$

thus for upper semi-continuous functions any of (2.5)-(2.7) can be used as a criterion for subharmonicity. By making use of the criterion '(2.6) for all harmonic h' Ransford proves that $u \in C^2(U)$ is subharmonic if and only if $h_{xx} + h_{yy} \geq 0$.

The following limit theorem for decreasing sequences of subharmonic functions is simple but useful.

Thm 26. *Let $(u_n)_n$ be subharmonic functions on an open set U in $\mathbb{C}$ and suppose that $u_1 \geq u_2 \geq u_3 \geq \ldots$ on U. Then $u := \lim_{n\to\infty} u_n$ is subharmonic.*

Proof. For each $\alpha \in \mathbb{R}$ the set $\{z : u(z) < \alpha\}$ is the union of the open sets $\{z : u_n(z) < \alpha\}$, so it is open too and hence u is upper semi-continuous.

If $\bar{\Delta}(w, \rho) \in U$ then the global submean inequality for u_n ensures

$$u_n(w) \leq \frac{1}{2\pi} \int_0^{2\pi} u_n(w + \rho e^{it}) dt \ .$$

for all n. Letting $n \to \infty$ and applying the monotone convergence theorem on deduces that u satisfies the submean inequality, hence is subharmonic. □

2.3 Integrability of subharmonic functions

Subharmonic functions are bounded above on compact sets but can be unbounded below. This poses the natural question about their integrability. The answer tuns out to be positive, subharmonic functions are locally integrable, implying that they cannot be 'too unbounded'.

Thm 27. *(Local Integrability) Let u be subharmonic on a domain D in $\mathbb{C}$ such that $u \neq -\infty$ identically on D. Then u is locally integrable.*

Proof. One has to verify that for every $w \in D$

$$\int_{\Delta(w,\rho)} |u(z)| d^2 z < \infty \quad \text{for some } \rho > 0 \;, \tag{2.8}$$

where $d^2z = dxdy$ is the element of area in the complex plane. Let A be the set of w possessing this property and B be its complement in D. The strategy of proof is to show that both A and B are open, and that $u = -\infty$ on B. This will imply that $D = A$ and $B = \emptyset$ since D is connected and $u \not\equiv -\infty$ on D.

A is open: Let $w \in A$ so that (2.8) holds. Then every $z \in \Delta(w,\rho)$ belongs to A as well. For, $\Delta(z,r) \subset \Delta(w,\rho)$ for sufficiently small r, and hence the integral of $|u|$ over $\Delta(z,r)$ is finite too.

B is open: Let $w \in B$ and choose ρ such that $\bar{\Delta}(w,2\rho) \subset D$. Then, because $w \in B$,

$$\int_{\Delta(w,\rho)} |u(z)|\, d^2 z = \infty \;.$$

Given $w' \in \Delta(w,\rho)$ set $\rho' = \rho + |w' - w|$. The disk about w' of radius ρ' covers $\Delta(w,\rho)$, hence

$$\int_{\Delta(w',\rho')} |u(z)|\, d^2 z = \infty \;.$$

As u is bounded above on compact sets, this implies that

$$\int_{\Delta(w',\rho')} u(z)\, d^2 z = -\infty \;.$$

By the submean inequality at w',

$$\pi \rho'^2 u(w') \le \int_0^{\rho'} \left(\int_0^{2\pi} u(w' + re^{it}) dt \right) r\, dr = \int_{\Delta(w',\rho')} u(z)\, d^2 z = -\infty \;.$$

(As u is bounded above on $\Delta(w',\rho')$, the repeated integral coincides with the double integral). Hence $u = infty$ on $\Delta(w,\rho)$. This implies that B is open and $u = -\infty$ on B. □

By the compactness argument, local integrability implies integrability on compact sets. This implies integrability on circles.

Corollary 28. *Let u subharmonic on a domain D and $u \not\equiv -\infty$ there. Then*

$$\int_0^{2\pi} u(w + \rho e^{it}) dt > -\infty \quad (\bar{\Delta}(w,\rho) \subset U) \;.$$

Proof. We may assume that $u \le 0$ on D (recall that u is bounded on compact sets). Then by the Poisson integral inequality, Thm 24, for any $r < \rho$,

$$\begin{aligned} u(w + re^{it}) &\le \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r\cos(\theta - t) + r^2} u(w + \rho e^{i\theta})d\theta \\ &\le \frac{\rho - r}{\rho + r}\frac{1}{2\pi}\int_0^{2\pi} u(w + \rho e^{i\theta})d\theta\ . \end{aligned}$$

Suppose that $\int_0^{2\pi} u(w + \rho e^{it})dt = -\infty$. Then the last inequality implies that $u = -\infty$ on $\Delta(w, \rho)$, and, consequently, u is not integrable. This contradicts Thm 27, hence the integral above is finite. □

By a standard measure theory argument, integrability on compact sets implies that the set of points in $\mathbb{C}$ where a subharmonic function takes value $-\infty$ has Lebesgue measure zero.

2.4 Three theorems not covered but useful (of course there are many more ....)

Thm 29. *Let U be an open subset of $\mathbb{C}$, and $u \in C^2(U)$. Then u is subharmonic on U if and only if $u_{xx} + u_{yy} \ge 0$.*

Thm 30. *(Liouville Theorem) Let u be subharmonic on $\mathbb{C}$ such that*

$$\limsup_{z\to\infty} \frac{u(z)}{\ln|z|} \le 0\ .$$

Then u is constant on $\mathbb{C}$. E.g., every subharmonic function on $\mathbb{C}$ which is bounded above must be constant.

Thm 31. *(Weak Identity Principle) Let u v be subharmonic on an open set U in $\mathbb{C}$. If $u = v$ almost everywhere on U then $u = v$ everywhere on $\mathbb{C}$. E.g., if two subharmonic functions coincide off the real line, they coincide on the real line too!*

2.5 Two topics not covered but important

There are strong similarities between subharmonic functions and convex functions on $\mathbb{R}$. More on this in Ransford's book, and also in Notions of Convexity by Lars Hörmander.

We also had no time to time to study the important technique of smoothing, mightn need to come back to this one later.

Exercises 2

(1) By making use of the integral

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|re^{it}-\zeta|\,dt = \begin{cases} \ln|\zeta|, & \text{if } r \le |\zeta|, \\ \ln r, & \text{if } r > |\zeta|, \end{cases}$$

show that the function $u(z) = \sum\limits_{n=1}^{\infty} 2^{-n}\ln|z-2^{-n}|$ is subharmonic.

(2a) Suppose that (u_n) is sequence of subharmonic functions on $\mathbb{C}$ such that $\sup_n u_n$ is bounded above on compact sets. Let $u = \sum_{n=1}^{\infty} \alpha_n u_n$, where $\alpha_n \ge 0$ and $\sum_{n=1}^{\infty} \alpha_n < \infty$. Prove that u is upper semi-continuous and satisfies the local submean inequality, hence, is subharmonic.

[Hint: Upper semi-continuity follows from Fatou's Lemma.]

(2b) Let (w_n) be a countable dense subset of the unit disk $\bar{\Delta}(0,1)$ in the complex plane. Define $u(z) = \sum_{n=1}^{\infty} 2^{-n}\ln|z-w_n|$. It follows from (2a) that u is subharmonic. Prove that u is discontinuous almost everywhere in $\bar{\Delta}(0,1)$.

[Hint: Consider the set $E = \{z : u(z) = -\infty\} \subset \mathbb{C}$. The function u is discontinuous at every point in $\bar{E}\backslash E$.]

3 Let u be a subharmonic function on $\Delta(0,1)$ such that $u < 0$. Prove that for every $\zeta \in \partial\Delta(0,1)$,

$$\limsup_{r\to 1^-} \frac{u(r\zeta)}{1-r} < 0$$

[Hint: Apply the maximum principle to $u(z) + c\ln|z|$ on the annulus $0.5 < |z| < 1$ for a suitable constant c.]

(Lecture 3, 5 March 2012)

3 Potentials and Generalised Laplacian

We shall only consider potentials of finite measures with compact support in $\mathbb{C}$. This captures the essence but avoids technicalities.

The measures we consider are Borel measures, i.e. positive measures on the Borel σ algebra of open sets in $\mathbb{C}$. The support $\operatorname{supp}\mu$ of measure μ is the (closed) set of all $z \in \mathbb{C}$ such that $\mu(\Delta(z,r)) > 0$ for any $r > 0$.

Definition (Logarithmic Potential) Let μ be a finite mass Borel measure on $\mathbb{C}$ with compact support. Its potential p_μ is the function

$$p_\mu(z) = \int_{\mathbb{C}} \ln|z-w|\, d\mu(w) \quad (z \in \mathbb{C}).$$

Thm 32. *p_μ is subharmonic on $\mathbb{C}$ and harmonic on $\mathbb{C}\backslash(\operatorname{supp}\mu)$.*

Proof. Let K be the support of μ. Since K is compact, the function $f_z(w) = \ln|z-w|$ is bounded above on K. It then follows that $p_\mu(z)$ is upper semi-continuous (by Fatou's Lemma), and has the submean property (by Fubini Theorem and subharmonicity of $\ln|z|$). Hence $p_\mu(z)$ is subharmonic on $\mathbb{C}$.

As K is closed, $\mathbb{C}\backslash K$ is open. Hence $p_\mu(z)$ is harmonic on $\mathbb{C}\backslash K$, e.g., by direct computation. □

Note that the log-potential is some time defined as $\int_{\mathbb{C}} \ln(1/|z-w|)\, d\mu(w)$, so that it is superharmonic function (lower semi-conitinuous and the inequality in the submean property in the other direction. This corresponds better to the real word, where the electric potential created by a point charge q at distance r from the charge is proportional to qr^{d-2} in d-dimensions word and to $q\ln(1/r)$ in two dimensions. Correspondingly, the potential due to a charge distribution $\rho(w)$ will be given by $\int_{\mathbb{C}} \ln(1/|z-w|)\, \rho(w) d^2w$.

Thm 33. *(Continuity at Boundary of Support) Let μ be a finite Borel measure with compact support K and $\zeta_0 \in \partial K$. Then*

$$\liminf_{z\to\zeta_0} p_\mu(z) = \liminf_{\zeta\to\zeta_0,\, \zeta\in K} p_\mu(\zeta)\ . \tag{3.1}$$

Furthermore,

$$\text{if} \quad \lim_{\zeta\to\zeta_0,\, \zeta\in K} p_\mu(\zeta) = p_\mu(\zeta_0) \quad \text{then} \quad \lim_{z\to\zeta_0} p_\mu(z) = p_\mu(\zeta_0)\ . \tag{3.2}$$

Proof. If $p_\mu(\zeta_0) = -\infty$ then $\lim_{z\to\zeta_0} p_\mu(z) = -\infty$ by upper semi-continuity, and (3.1) holds.

Suppose now that $p_\mu(\zeta_0) > -\infty$. Then $\mu(\{\zeta_0\}) = 0$. Since $\mu(\cap_n A_n) = \lim_{n\to\infty} \mu(A_n)$, it then follows that for every $\epsilon > 0$ there exists $r > 0$ such that $\mu(\Delta(\zeta_0, r)) < \epsilon$. By definition of lim sup,

$$\liminf_{z\to\zeta_0} p_\mu(z) \le \liminf_{\zeta\to\zeta_0,\ \zeta\in K} p_\mu(\zeta) ,$$

so to prove (3.1) it will suffice to prove the inequality in the opposite direction.

For every $z \in \mathbb{C}$ choose $\zeta_z \in K$ that minimizes $|z - \zeta|$ over $\zeta \in K$. Then

$$\frac{|\zeta_z - w|}{|z - w|} \le \frac{|z - \zeta_z| + |z - w|}{|z - w|} \le 2 .$$

Therefore

$$p_\mu(z) \ge p_\mu(\zeta_z) - \epsilon \ln 2 - \int_{K\setminus\Delta(\zeta_0,r)} \ln \frac{|\zeta_z - w|}{|z - w|}\, d\mu(w) .$$

As $z \to \zeta_0$ in $\mathbb{C}$, the corresponding $\zeta_z \to \zeta_0$ in K, and, hence,

$$\liminf_{z\to\zeta_0} p_\mu(z) \ge \liminf_{\zeta_z\to\zeta_0,\ \zeta\in K} p_\mu(\zeta_z) - \epsilon \ln 2 .$$

Be letting $\epsilon \to 0$,

$$\liminf_{z\to\zeta_0} p_\mu(z) \ge \liminf_{\zeta_z\to\zeta_0,\ \zeta\in K} p_\mu(\zeta_z) \ge \liminf_{\zeta\to\zeta_0,\ \zeta\in K} p_\mu(\zeta) ,$$

which proves (3.1). In its turn, (3.1) and upper semi-continuity proves (3.2). □

Thm 34. *(Minimum Principle) Let μ be a finite Borel measure with compact support K.*

If $p_\mu \ge M$ on K then $p_\mu \ge M$ on $\mathbb{C}$.

Proof. The function $u = -p_\mu(z)$ is subharmonic on $\mathbb{C}\setminus K$ and $u(z) \to -\infty$ as $z \to \infty$. Then for all sufficiently large $R > 0$, $u(R) \le -M$. Applying the maximum principle for subharmonic functions to u and the domain $D = \Delta(0, R)\setminus K$, one concludes that $u(z) \le -M$ on D for all sufficiently large R, hence the theorem. □

3.1 Polar Sets

Polar sets play the role of negligible sets in potential theory. To define polar sets, we need first to introduce the concept of energy associated with measure (actually, anti-energy, due to our choice of sign of the logarithmic potential, so that minimisation of physical energy would correspond to maximisation of $I(\mu)$ as defined below.)

Definition Let μ be a finite Borel measure with compact support. Its energy $I(\mu)$ is defined as

$$I(\mu) = \int p_\mu d\mu = \int \int \ln |z - w| d\mu(z) d\mu(w).$$

Since p_μ is bounded above on compact sets, the above integral is well defined, and $I(\mu) < \mu(K) \sup_K p_\mu$. However, it may happen that $I(\mu) = -\infty$. For example, the Dirac measure $d\mu = \delta_{z_0}$ has potential $\ln |z - z_0|$, hence $I(\delta_{z_0}) = -\infty$. Similarly, any measure supported by a finite or countable set has infinite energy. Sets supporting only measures of infinite energy, of which a countable set is an example, are called polar sets.

Definition (Polar Set) A subset E of $\mathbb{C}$ is called polar if $I(\mu) = -\infty$ for every non-zero Borel measure μ of finite mass supported on a compact subset of E.

The theorem below asserts that, in the language of electrostatics, bounded Borel polar can hold no charge.

Thm 35. *Let μ be a finite Borel measure with compact support and such that $I(\mu) > -\infty$. Then $\mu(E) = 0$ for every Borel polar set E.*

Proof. Let E be a measurable set such that $\mu(E) > 0$. By regularity of μ, one can choose a compact subset $K \subset E$ such that $\mu(K) > 0$. Set $\tilde{\mu} = \mu|_K$. Then $\tilde{\mu}$ is a finite measure with support K. Set now d to be the diameter of the support of μ, so that $\ln \frac{|z-w|}{d} \leq 0$ for all $z, w \in \operatorname{supp} \mu$. Then

$$\begin{aligned} I(\tilde{\mu}) &= \int_K \int_K \ln \frac{|z-w|}{d} d\mu(z) d\mu(w) + \mu(K)^2 \ln d \\ &\geq \int_{\mathbb{C}} \int_{\mathbb{C}} \ln \frac{|z-w|}{d} d\mu(z) d\mu(w) + \mu(K)^2 \ln d \\ &= I(\mu) - \mu(\mathbb{C})^2 \ln d + \mu(K)^2 \ln d > -\infty \,, \end{aligned}$$

so E cannot be polar. □

Corollary 36. *Every Borel polar set in $\mathbb{C}$ has 2D Lebesgue measure zero.*

Proof. Let $R > 0$ and $\tilde{\mu}$ be the restriction the two-dimensional Lebesgue measure to the disk $\Delta(0, R)$. Since $\ln|z|$ is locally integrable, its potential $p_{\tilde{\mu}}(z) = \int_{\Delta(0,R)} \ln|z-w|d^2w > -\infty$, hence integrable on $\Delta(0, R)$ by the local integrability theorem for subharmonic functions. This can also be checked directly of course. Therefore, $I(\tilde{\mu}) > -\infty$ for every R. The assertion of Corollary now follows from Thm 35 by letting $R \to \infty$. □

A property is said to hold nearly everywhere (n.e.) on $S \subset \mathbb{C}$ if it holds on $S \backslash E$ for some Borel polar E. Thus 'nearly everywhere' implies almost everywhere. The converse is false (e.g., the unit interval [0,1] has Lebesgue measure zero but is not polar).

Corollary 37. *A countable union of Borel polar sets is polar too.*

Proof. Let $E = \cup_n E_n$ and E_n are all polar. Suppose that μ is a finite non-zero Borel measure supported by $K \subset E$. Suppose $I(\mu) > -\infty$. Then, by Thm 35, $\mu(E_n) = 0$ for all n. Hence, $\mu(E) = 0$ too, and μ must be zero as $\mu(K) \leq \mu(E) = 0$. Therefore, $I(\mu) = -\infty$ for $\mu \neq 0$, and E is polar. □

3.2 Equilibrium Measure

Definition (Equilibrium Measure) Let K be a compact subset of $\mathbb{C}$ and denote by $\mathcal{P}(K)$ the set of all Borel probability measures on K. If there exists $\nu \in \mathcal{P}(K)$ such that

$$I(\nu) = \sup_{\mu \in \mathcal{P}(K)} I(\mu)$$

then this measure ν is called the equilibrium measure.

Thm 38. *Every compact subset K of $\mathbb{C}$ has an equilibrium measure. If K is non-polar this equilibrium measure is unique.*

Proof. We shall only prove existence. Set $M = \sup_{\mu \in \mathcal{P}(K)} I(\mu)$ and choose a sequence $(\mu_n)_n$ in $\mathcal{P}(K)$ such that $I(\mu_n) \to M$ as $n \to \infty$. By Helly's selection theorem, $(\mu_n)_n$ contains a weakly converging subsequence (μ_{n_k}), i.e. for any bounded continuous function f,

$$\lim_{k\to\infty} \int_K f(z) d\mu_{n_k}(z) = \int_K f(z) d\mu(z) \; . \tag{3.3}$$

The limiting measure μ is of course a probability measure.

The Stone-Weierstrass theorem asserts that every continuous function of two variables $\chi(z,w)$ on $K \times K$ can be uniformly approximated by finite sums of the form $\sum_j \phi_j(z)\psi_j(w)$, where the ϕ_j's and ψ_j's are continuous on K. Therefore(3.3) implies

$$\lim_{k\to\infty} \int_K \int_K \chi(z,w)d\mu_{n_k}(z)d\mu_{n_k}(w) = \int_K \int_K \chi(z,w)d\mu(z)d\mu(w) \ . \qquad (3.4)$$

This can be applied to the energy functional $I(\mu)$ after introducing a regularisation the log function,

$$f_\epsilon(z) = \begin{cases} \ln|z|, & \text{if } |z| \geq \epsilon; \\ \ln\epsilon, & \text{if } |z| < \epsilon. \end{cases}$$

Obviously, $\ln|z| \leq f_\epsilon(z)$, so that

$$I(\mu_{n_k}) \leq \int_K \int_K f_\epsilon(z-w)d\mu_{n_k}(z)d\mu_{n_k}(w) \ .$$

Now it follows from (3.4) that

$$M = \lim_{k\to\infty} I(\mu_{n_k}) \leq \int_K \int_K f_\epsilon(z-w)d\mu(z)d\mu(w) \ .$$

Hence, by the monotone convergence theorem, $M \leq I(\mu)$, and $M = I(\mu)$. □

Thm 39. *(Frostman's Theorem) Let K be a compact set in $\mathbb{C}$, and let ν be an equilibrium measure for K. Then:*

(a) $p_\nu \geq I(\nu)$ on $\mathbb{C}$.

(b) $p_\nu = I(\nu)$ on $K \backslash E$, where E is a polar subset of ∂K.

There are several proofs of this important theorem, one can be in Ransford's book.

3.3 The Generalised Laplacian and Poisson's Equation

Laplacian: $\Delta u = u_{xx} + u_{yy}$.

Thm 40. *Let U be an open subset of $\mathbb{C}$, and $u \in C^2(U)$. Then u is subharmonic on U if and only if $\Delta u \geq 0$.*

Proof. Let $u \in C^2(U)$ and assume that $\Delta u \geq 0$. To show that u is subharmonic it will suffice to verify the harmonic majoration criterion (2.6) (see the corresponding remark following proof of the Global Submean Inequality).

Correspondingly, let D be a relatively compact subdomain of U and h be harmonic on D and such that $\limsup_{z\to\zeta}(u-h)(z) \leq 0$ for all $\zeta \in \partial D$. For $\epsilon > 0$, define

$$v_\epsilon(z) = \begin{cases} u(z) - h(z) + \epsilon|z|^2, & \text{if } z \in D \\ \epsilon|z|^2, & \text{if } z \in \partial D \;. \end{cases}$$

Then v_ϵ is upper semi-continuous on $\bar{D}$, and, hence, attains a maximum there. It cannot attain a local maximum on D because $\Delta v_\epsilon \geq 4\epsilon > 0$, hence the maximum is attained at the boundary ∂D. This implies that $u(z) - h(z) + \epsilon|z|^2 \leq \epsilon \max_{z\in\partial D}|z|^2$ on D for any $\epsilon > 0$, hence $u - h \leq 0$ on D. Hence u is subharmonic.

Suppose now that u is subharmonic and $\Delta u < 0$ at $w \in D$. By continuity, $\Delta u < 0$ in a neighbourhood of w, and therefore, by the above argument, u must be superharmonic there, which is impossible. □

This theorem can be generalised to arbitrary subharmonic functions if the Laplacian is understood in the sense of distributions. To this end, it is necessary to make a short excursion into distribution theory. Our starting point is Green's theorem. Let D be a domain in the complex plane. Then Green's theorem asserts that under appropriate conditions on ϕ, ψ,

$$\int_D (\phi\Delta u - u\Delta\phi)d^2z = \oint_{\partial D} (u\frac{\partial\phi}{\partial n} - \phi\frac{\partial u}{\partial n})dS \;, \tag{3.5}$$

where $\frac{\partial}{\partial n}$ is the directional derivative along the inward normal into D. In particular, if $\phi \in C_c^\infty(D)$, the set of all C^∞ functions whose support is a compact subset of D, and u is C^2 subharmonic then

$$\int_D \phi\Delta u\, dz^2 = \int_D u\Delta\phi\, d^2z \;.$$

In view of Thm 40, $\Delta u\, dz^2$ can be identified with a positive measure. This measure is normally denoted by Δu, so that

$$\int_D \phi\Delta u \;= \int_D u\Delta\phi\, d^2z \quad (\phi \in C_c^\infty(D)) \;. \tag{3.6}$$

The right hand-side above makes sense for arbitrary subharmonic functions which are not identically $-\infty$ (as such u are locally integrable). The theorem below asserts that it actually defines a Radon measure which is known as the generalised Laplacian. Radon measures are Borel measures with the property that the total mass of every compact set is finite.

Thm 41. *Let u be a subharmonic function on a domain D in $\mathbb{C}$ and $u \not\equiv -\infty$. Then there exists a unique Radon measure Δu such that (3.6) holds.*

Proof. Since u is locally integrable, the integral on the right hand-side in (3.6) defines a linear functional Λ_u on $C_c^\infty(D)$,

$$\Lambda_u \phi = \int_D u \, \Delta\phi \, d^2 z \quad (\phi \in C_c^\infty(D)) \ .$$

It will suffice to prove that Λ_u is positive and can be extended, by continuity, onto the space $C_c(D)$ of continuous functions with compact support in D. Then the existence of a positive Radon measure μ, such that $\Lambda\phi = \int_D \phi \, d\mu$, and its uniqueness, will follow from the Riesz representation theorem.

Step 1 : Positivity on $C_c^\infty(D)$.

Let $\phi \in C_c^\infty(D)$ with $\phi \geq 0$. Choose a relatively compact set U in D that covers the support of ϕ. By employing the standard technique of smoothing by convolutions (for details see, e.g., Ransford's book), given a subharmonic function u on U, with $u \not\equiv -\infty$, there exist C^∞ subharmonic functions u_n such that $u_n \downarrow u$. By Thm 40 $\Delta u_n \geq 0$. Therefore

$$\int u_n \, \Delta\phi \, d^2 z = \int \phi \, \Delta u_n \, d^2 z \geq 0 \ .$$

We have $u \leq u_n \leq u_1$, for all n, with u being locally integrable and u_1 bounded. Letting $n \to \infty$ in the above inequality, we conclude that $\Lambda_u \phi \geq 0$ by the dominated convergence theorem.

Step 2: Extending Λ_u from $C_c^\infty(D)$ to $C_c(D)$.

Let $\phi \in C_c(D)$ and U be a relatively compact set in D covering the support of ϕ. By employing the technique of smoothing by convolutions, ϕ can be approximated in the uniform norm by $\phi_n \in C_c^\infty(D)$, so that $||\phi - \phi_n||_\infty$ can be made arbitrary small. The ϕ_n's are supported inside U and if $\phi \geq 0$ then so are ϕ_n, see Ransford's book for details. Now, choose $\psi \in C_c^\infty(D)$ such that $\psi = 1$ on U and $0 \leq \psi \leq 1$ throughout D and set $C = L_u \psi$. By positivity of L_u, $|L_u \phi_n| \leq C||\phi_n||_\infty$, hence $L_u \phi_n$ has a limit as $n \to \infty$, which we will assign to be $L_u \phi$. This gives the desired extension of L_u to $C_c^\infty(D)$ to $C_c(D)$. The extended functional is positive by continuity.

Step 3: Uniqueness.

Suppose that μ_1 and μ_2 are two Radom measures such that $\int_D \phi\, d\mu_1 = \int_D \phi\, d\mu_2$ for all $\phi C_c^\infty(D)$. Since any function in $C_c(D)$ can be approximated by functions from $C_c^\infty(D)$ as above, the two integrals are also equal for test functions from $C_c(D)$. Hence, by the uniqueness part of the Riesz representation theorem $\mu_1 = \mu_2$. □

The following theorem allows one to restore the measure from its potential and is fundamental.

Thm 42. *(Poisson's Equation) Let μ be a finite Borel measure on $\mathbb{C}$ with compact support. Then*

$$\Delta p_\mu = 2\pi\, \mu\ .$$

Proof. By definition of the generalised Laplacian (3.6), we have to show that

$$\int_{\mathbb{C}} p_\mu\, \Delta\phi\, d^2z = \int_{\mathbb{C}} 2\pi\phi\, d\mu \quad (\phi \in C_c^\infty(D))\ .$$

Correspondingly, let $\phi \in C_c^\infty(D)$. Then

$$\int_{\mathbb{C}} p_\mu\, \Delta\phi\, d^2z = \int_{\mathbb{C}} \left(\int_{\mathbb{C}} \ln|z-w|\, \Delta\phi(z)\, d^2z \right) d\mu(w)\ ,$$

where we have used Fubini's theorem. This is justified as $\ln|z|$ is locally integrable and ϕ has a compact support (and bounded). Now for fixed w, the function $\ln|\cdot - w|$ is harmonic away from w. On making use of Green's theorem (3.6)

$$\begin{aligned}
\int_{\mathbb{C}} \ln|z-w|\, \Delta\phi(z)\, d^2z &= \lim_{\epsilon\to 0} \int_{|z-w|>\epsilon} \ln|z-w|\, \Delta\phi(z)\, d^2z \\
&= \lim_{\epsilon\to 0} \int_0^{2\pi} \left(\phi(w+\epsilon e^{it}) - \epsilon\ln\epsilon\, \frac{\partial\phi}{\partial r}(w+re^{it})\,|_{r=\epsilon} \right) dt \\
&= 2\pi\phi(w)\ ,
\end{aligned}$$

where, to arrive at the integral in the middle by Green's theorem, we have used that (i) the arc-length of the circle of radius r is the arc-length of the unit circle times r, and (ii) the corresponding inner normal is $-(x/r, y/r)$ so that $\frac{\partial}{\partial n} \ln|z| = -\frac{1}{r}$ on the circle $|z| = r$. □

Corollary 43. *Let μ_1 and μ_2 be finite Borel measures on $\mathbb{C}$ with compact support. If $p_{\mu_1} = p_{\mu_2} + h$ on an open set U, where h is harmonic on U, then $d\mu_1\,|_U = d\mu_2\,|_U$.*

The converse is also true.

Lemma 44. *(Weyl's Lemma) Let u and v be subharmonic functions on a domain D in $\mathbb{C}$, with $u, v \neq -\infty$. If $\Delta u = \Delta v$ then $u = v + h$ where is harmonic on D.*

For proof, see e.g. Ransford book.

Weyl's Lemma is important because it implies that any non-trivial subharmonic function can be written as the sum of log-potential and a harmonic function. The corresponding statement is known as the Riesz decomposition theorem.

Thm 45. *(Riesz Decomposition Theorem) Let u be a subharmonic function on a domain D in $\mathbb{C}$, with $u \neq -\infty$. Given a relatively compact open subset U of D,*

$$u = p_\mu + h \quad \text{on } U \ ,$$

where $\mu = \frac{1}{2\pi}\Delta u\,|_U$ and h is harmonic on U.

Proof. Set $\mu = \frac{1}{2\pi}\Delta u\,|_U$ Then $\Delta p_\mu = 2\pi\mu = \Delta u$ on U. The result now follows from Weyl's lemma (applying it on each component of U). □

The following a straightforward application of Corollary 43 (Poisson's equation to be precise).

Thm 46. *Let f be holomorphic on a bounded domain D and is not identically zero there, and let μ be the zero counting measure for f, in the sense that it assigns mass 1 to each of zeros of f, counted according to multiplicities. Then $\mu = \frac{1}{2\pi}\Delta \ln|f|$.*

Proof. Let $(z_n)_{n=1}^N$ be zeros of f in D. Then $f(z) = g(z)\prod_{jn=1}^N (z - z_n)$ for some holomorphic g which is non-zero D, and

$$\ln|f(z)| = \sum_{n=1}^{N} \ln|z - z_n| + \ln|g(z)| \ .$$

The first term above is the potential of μ and the second term is a harmonic function. Hence, by Poisson's equation, $\mu = \frac{1}{2\pi}\Delta \ln|f|$. □

Obviously, the relation $\mu = \frac{1}{2\pi}\Delta \ln|f|$ extends to unbounded domains.

(Lecture 4, 12 March 2012)

3.4 Poisson's Equation, Continued

Poisson's equation relates measures and their potentials. This can be exploited to obtain limiting distributions of roots of polynomials or eigenvalues of matrices in the limit of large degree/matrix dimension by the way of calculating the limiting potential.

Let $z_1, \ldots, z_n$ be n points in a bounded domain $\mathbb{C}$, not necessarily distinct, and denote by μ_n the unit mass measure that assigns mass $\frac{1}{n}$ to each of z_j, i.e.

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{z_j},$$

where δ_z is the Dirac measure supported by $\{z\}$. We shall call μ_n the normalised counting measure. The (log)-potential of μ_n is

$$p_n(z) = \int \ln |z - w| \, d\mu_n(w) = \frac{1}{n} \sum_{j=1}^{n} \ln |z - z_j| \, .$$

and correspondingly $\frac{1}{2\pi} \Delta p_n = \mu_n$. The theorem below asserts that this relation holds in the limit $n \to \infty$

Thm 47. *(Widom's Lemma) Suppose that the counting measures μ_n have all support inside a bounded domain in $\mathbb{C}$. If p_n converges to p as $n \to \infty$ almost everywhere (with respect to the Lebesgue measure) in $\mathbb{C}$ then p is locally integrable, $\Delta p \geq 0$ and the measures μ_n converge weakly to the measure $\mu = \frac{1}{2\pi} \Delta p$.*

Proof. Local integrability of $\ln |z|$ implies that the family of functions (p_n) is uniformly integrable on compact sets with respect to the Lebesgue measure on $\mathbb{C}$. That is, for every compact K and every $\epsilon > 0$ there exists a $\delta > 0$ such that if B is a subset of K of measure less than δ than $\int_B |f_n| d^2 < \epsilon$ for every n. It follows from the uniform integrability and Egorov's theorem that for any continuous function ψ with compact support

$$\lim_{n \to \infty} \int p_n(z) \psi(z) \, d^2 z = \int p(z) \psi(z) \, d^2 z \, .$$

In particular, p is locally integrable and Δp is well defined (as a distribution). As $\Delta p_n \geq 0$ then so is Δp by limiting transition, and thus is a measure (see Step 2 in proof of Thm 41). For $\phi \in C_c^\infty$),

$$\int \phi \, \Delta p \; = \int p \, \Delta \, \phi d^2 z = \lim_n \int p_n \, \Delta \phi \, d^2 z = \lim_n \int \phi \, \Delta p_n \, .$$

which means that the measures μ_n converge to μ as distributions. As any sequence of measures of unit mass (supported inside a compact set) that converge as distributions must converge weakly, the result follows. $\square$

With a little bit more effort it can be shown that $p(z) = \int \ln|z-w|d\mu(w)$ almost everywhere.

Consider now an application of this theorem to the problem of finding the limiting distribution of zeros of truncated exponential series $\sum_{j=0}^{n} \frac{w^j}{j!}$ in the limit $n \to \infty$. Anticipating that zeros might spread in the complex plane, we scale w with n by introducing a new variable $z = w/n$. Thus consider

$$f_n(z) = \frac{n!}{n^n} \sum_{j=0}^{n} \frac{(nz)^j}{j!} \ .$$

The factor in front of the series is introduced for convenience, so that f_n is a monic polynomial of degree n, $f_n(z) = z^n + \ldots$. Correspondingly, the normalised counting measure of its zeros $z_1, \ldots, z_n$ is

$$p_n(z) = \frac{1}{n} \sum_{j=1}^{n} \ln|z - z_j| = \frac{1}{n} \ln|f_n(z)| \ .$$

A simple estimate shows that $f_n(z)$ has no zeros outside the disk $|z| \leq 2$ for any n. Indeed, suppose that z_0 is a zero of $f_n(z)$ and $|z_0| > 1$. Then

$$\begin{aligned} \frac{n^n|z_0|^n}{n!} \leq \sum_{j=0}^{n-1} \frac{n^j|z_0|^j}{j!} &\leq \frac{n^{n-1}|z_0|^{n-1}}{(n-1)!} \sum_{j=0}^{n-1} \frac{(n-1)!}{(n-1-j)!n^j} \frac{1}{|z_0|^j} \\ &\leq \frac{n^{n-1}|z_0|^{n-1}}{(n-1)!} \sum_{j=0}^{n-1} \frac{1}{|z_0|^j} \leq \frac{n^{n-1}|z_0|^{n-1}}{(n-1)!} \frac{|z_0|}{|z_0|-1} \ . \end{aligned}$$

It follows from this that $|z_0| \leq 2$.

Thus the zero counting measure μ_n has support inside the disk $|z| \leq 2$ for all n. Our strategy will be to evaluate the potential p_n of μ_n in the limit $n \to \infty$ and then apply Widom's lemma.

Note the following identity (which is easy to verify by binomial expansion (recall Euler's integral $n! = \int_0^\infty s^n e^{-s}\, ds$),

$$\sum_{j=0}^{n} \frac{x^j}{j!} = \frac{1}{n!} \int_0^\infty (x+s)^n e^{-s}\, ds = \frac{e^x}{n!} \int_x^\infty t^n e^{-t}\, dt \ ,$$

It follows from it that

$$f_n(z) = ne^{nz} \int_z^{+\infty} s^n e^{-ns}\, ds = ne^{nz} \int_z^{+\infty} e^{n(\ln s - s)}\, ds\ .$$

This is in a form convenient for a saddle point analysis (steepest descent) of the integral $\int_z^{+\infty} \exp(n(\ln s - s))\, ds$. The saddle point equation is the equation $(\ln s - s)' = 0$ and there is only one solution $s_0 = 1$. If $\mathrm{Re}(\ln z - z) < -1$ and $\mathrm{Re}\, z < 1$ the integration path can be deformed to pass through the saddle point and the integral is dominated by a neighbourhood s_0. This gives

$$\int_z^{+\infty} e^{n(\ln s - s)}\, ds \sim \sqrt{\frac{2\pi}{n}} e^{n(\ln s_0 - s_0)} = \sqrt{\frac{2\pi}{n}} e^{-n}\ .$$

On the other side if $\mathrm{Re}(\ln z - z) > -1$ or $\mathrm{Re}\, z > 1$ then the integration path cannot be deformed to path through $s_0 = 1$ and the integral is dominated by the end point $s = z$ of the integration path. This gives

$$\int_z^{+\infty} e^{n(\ln s - s)}\, ds \sim \frac{1}{n} e^{n(\ln z - z)}\ .$$

Collecting these two results together,

$$f_n(z) \sim \begin{cases} \sqrt{2\pi n} e^{n(z-1)} & \text{if } \ln|z| - \mathrm{Re}\, z < -1 \text{ and } \mathrm{Re}\, z < 1 \\ \frac{1}{n} e^{n \ln z} & \text{if } \ln|z| - \mathrm{Re}\, z > -1 \text{ or } \mathrm{Re}\, z > 1 \end{cases}$$

The equation $\ln|z| - \mathrm{Re}\, z = -1$,or equivalently, $|ze^{1-z}| = 1$, defines a curve which is symmetric about the real axis and intersects itself once at $z = 1$. The part of this curve in the half plane $\mathrm{Re}\, z \leq 1$ is called the Szegö curve. It is a closed curve and its interior is the set of points in the half plane $\mathrm{Re}\, z \leq 1$ where $|ze^{1-z}| < 1$.

From the above asymptotic formula for f_n it is apparent that the limit of $p_n(z) = \frac{1}{n} \ln|f_n(z)|$ exists everywhere off the Szegö curve:

$$p(z) = \lim_{n\to\infty} \frac{1}{n} \ln|f_n(z)| = \begin{cases} \mathrm{Re}\, z - 1 & \text{inside the Szegö curve} \\ \ln|z| & \text{outside the Szegö curve} \end{cases}$$

The Szegö curve has measure zero, so we have established a.e. convergence of the potentials. Applying Widom's lemma, the normalised zero counting measure of truncated exponential series weaklyy converges to $\mu = \frac{1}{2\pi}\Delta p$. As p is harmonic everywhere except on the Szegö curve, we conclude that the limiting counting measure of scaled zeros of the truncated exponential series is supported by the Szegö curve. The density of the distribution of zeros with respect of the arc-length can be found by evaluating the jumpo in the normal derivative of the potential across the curve.

4 The Dirichlet Problem and Harmonic Measure

Let D be a bounded domain of $\mathbb{C}$, and $\phi : \partial D \to \mathbb{R}$ be a continuous function. The associated Dirichlet problem is to find a function h such that

$$\Delta h = 0 \qquad \text{(on } D) \tag{4.1}$$
$$\lim_{z\to\zeta} h(z) = \phi(\zeta) \quad (\text{for every } \zeta \in \partial D)\ . \tag{4.2}$$

Of course, the Dirichlet problem makes sense for unbounded subdomains of $\mathbb{C}$ as well (and for other differential operators) but we shall focus on bounded domains here.

The Dirichlet problem (for bounded domains) can have no more than one solution. This follows from the maximum principle for harmonic functions. Indeed, if h_1 and h_2 are solutions then $h = h_1 - h_2$ is harmonic on D and $h = 0$ on ∂D, hence $h_1 \le h_2$ on D. Reversing the order of h_1 and h_2, one obtains $h_1 \ge h_2$, hence $h_1 = h_2$. For unbounded domains one can have more than one solution, e.g. $h = \operatorname{Re} z$ is harmonic on the right half of the complex plane $\operatorname{Re} z \ge 0$.

4.1 Perron Function

Definition (Perron Function) Let D be a bounded domain of $\mathbb{C}$ and $\phi : \partial D \to \mathbb{R}$ be a bounded function. The associated Perron function $H_D\phi : D \to \mathbb{R}$ is defined as

$$H_D\phi = \sup_{u\in\mathcal{U}} u$$

where the supremum is taken over the set $\mathcal{U}$ of all subharmonic on D functions u such that $\limsup_{z\to\zeta} u(z) \le \phi(\zeta)$ for every $\zeta \in \partial D$.

The Perron function is harmonic on D. In order to prove it we shall need two technical lemmas.

Lemma 48. *(Glueing Theorem) Let U and V be two open sets in $\mathbb{C}$, $V \subset U$. Suppose that u and v are subharmonic on U and V, respectively, and $\limsup_{z\to\zeta} v(z) \le u(\zeta)$ for every $\zeta \in U \cap \partial V$. Then the function*

$$\tilde{u} = \begin{cases} \max(u, v) & \text{on } V; \\ u & \text{on } U \backslash V \end{cases}$$

is subharmonic on U.

Proof. The condition $\limsup_{z\to\zeta} v(z) \le u(\zeta)$ ensures that $\tilde{u}$ is upper semi-continuous. One can easily check the local submean property on V, so $\tilde{u}$ is subharmonic on V. If $w \in U\backslash V$ then $\tilde{u}(w) = u(w)$ and hence $\tilde{u}(w) \le \frac{1}{2\pi}\int_0^{2\pi} u(w+re^{it})\,dt$ by the local submean property for u. As $u \le \tilde{u}$, the local submean property for $\tilde{u}$ follows. □

We defined the Poisson integral

$$P_\Delta\phi(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - |z-w|^2}{|z-w-\rho e^{i\theta}|^2}\,\phi(w+\rho e^{i\theta})d\theta \quad (z \in \Delta),$$

with $\Delta = \Delta(w,\rho)$, for continuous functions ϕ. Of course, if ϕ is only integrable then the integral above still makes sense and is a harmonic function on Δ (as the real part of a holomorphic function). In particular, if u is subharmonic on a neighbourhood of Δ and not identically $-\infty$ then $u(w+\rho e^{it})$ is integrable, and, hence, $P_\Delta u$ is harmonic on Δ. With this observation in hand:

Lemma 49. *(Poisson Modification) Let D be a bounded domain in $\mathbb{C}$ and Δ be an open disk about a point in D such that $\bar{\Delta} \subset D$. Suppose that u is subharmonic on D with $u \not\equiv -\infty$. Define the function*

$$\tilde{u} = \begin{cases} P_\Delta u & \text{on } \Delta; \\ u & \text{on } D\backslash\Delta. \end{cases}$$

Then $\tilde{u}$ is subharmonic on D, harmonic on Δ and $\tilde{u} \ge u$ on D.

Proof. $P_\Delta u$ is harmonic on Δ, and, by the Poisson integral inequality, Thm 24, $u \le P_\Delta u$ there, hence $u \le \tilde{u}$ on D.

Since u is upper semi-continuous, by Thm 20 we can find continuous functions ψ_n such that $\psi_n \downarrow u$ on $\partial\Delta$ as $n \to \infty$. Then for every $\zeta \in \partial\Delta$,

$$\limsup_{z\to\zeta} P_\Delta u(z) \le \lim_{z\to\zeta} P_\Delta \psi_n(z) = \psi_n(\zeta)\ .$$

By letting $n \to \infty$, $\limsup_{z\to\zeta} P_\Delta u(z) \le u(\zeta)$. We can now apply the Glueing theorem with $v = P_\Delta u$ which proves that $\tilde{u}$ is subharmonic. □

Now we can prove the key fact about the Perron function.

Thm 50. *Let D be a bounded domain of $\mathbb{C}$ and $\phi : \partial D \to \mathbb{R}$ be a bounded function. Then the Perron function $H_D\phi$ is harmonic on D and*

$$\sup_D |H_D\phi| \le \sup_{\partial D} \phi\ .$$

Proof. The desired inequality is straightforward. Let $M = \sup_D |\phi|$ and $\mathcal{U}$ be the set of subharmonic functions on D such that $\limsup_{z\to\zeta} u(z) \le \phi(\zeta)$ for $\zeta \in \partial D$. If $u \in \mathcal{U}$, then $\limsup_{z\to\zeta} u(z) \le M$, and, by the maximum principle, $u \le M$ on D. Hence $H_D\phi \le M$. On the other hand, the constant function $-M$ is in $\mathcal{U}$. Therefore $-M \le H_D\phi$. Thus $\sup_D |H_d\phi| \le M$.

It remains to show that $H_D\phi$ is harmonic. Since this property local, it will suffice to prove that $H_D\phi$ is harmonic on any open disk Δ with $\bar{\Delta} \subset D$. Fix such a Δ.

Let $w_0 \in \Delta$. By definition of $H_D\phi$, there exists a sequence (u_n) of functions in $\mathcal{U}$ such that $u_n(w_0) \to H_D\phi(w_0)$. Replacing u_n by $\max(u_1, \ldots, u_n)$, we may suppose that (u_n) is non-decreasing[3] Now applying the Poisson modification to each of u_n to obtain a non-decreasing sequence $\tilde{u}_n$ (the Poisson kernel is positive, hence $P_\Delta u_n$ is non-decreasing). Let $\tilde{u} = \lim_{n\to\infty} \tilde{u}_n$. Then:

(i) $\tilde{u} \le H_D\phi$ on D. Indeed, each $\tilde{u}_n$ is subharmonic on D. Since $\tilde{u}_n = u_n$ sufficiently close to the boundary of D, it follows that $\tilde{u}_n \in \mathcal{U}$, and, hence, $\tilde{u}_n \le H_D\phi$ and so is the limit.

(ii) $\tilde{u}(w_0) = H_D\phi(w_0)$. Indeed, by the Poisson modification lemma $\tilde{u}_n(w_0) \ge u_n(w_0)$. As $u_n(w_0) \to H_D\phi(w_0)$, we have that $\tilde{u}(w_0) \ge H_D\phi(w_0)$. And by (i) $\tilde{u}(w_0) \le H_D\phi(w_0)$.

(iii) $\tilde{u}$ is harmonic on Δ (as $(\tilde{u}_n)$ non-decreasing and each $\tilde{u}_n$ harmonic on Δ (Harnack's theorem).

Now we shall prove that $\tilde{u} = H_D\phi$ on Δ. To this end, fix an arbitrary point w and choose $(v_n) \in \mathcal{U}$ such that $v_n(w) \to H_D\phi(w)$. Replacing v_n by $\max(u_1, \ldots, u_n, v_1, \ldots, v_n)$, we may suppose that (v_n) is non-decreasing and $v_n \ge u_n$. Let $\tilde{v}_n$ be the Poisson modification of v_n. Then as before

(i) $\tilde{v} \le H_D\phi$ on D.

(ii) $\tilde{v}(w) = H_D\phi(w)$.

(iii) $\tilde{v}$ is harmonic on Δ

It follows from (i) that $\tilde{v}(w_0) \le H_D\phi(w_0) = \tilde{u}(w_0)$. On the other hand, $\tilde{v}_n \ge \tilde{u}_n$ for all n, hence $\tilde{v} \ge \tilde{u}$. Thus the function $\tilde{u} - \tilde{v}$, which is harmonic on Δ, attains a maximum value 0 at w_0, which means, by the maximum principle that $\tilde{v} = \tilde{u}$ on Δ. By (ii), $\tilde{v}(w) = H_D\phi(w)$, and so $\tilde{u}(w) = H_D\phi(w)$. As w is arbitrary, $\tilde{u} = H_D\phi$ on Δ, and as $\tilde{u}$ is harmonic on Δ, so is $H_D\phi$. □

[3]Obviously $f_n = \max(u_1, \ldots, u_n) \in \mathcal{U}$, hence $f_n(w) \le H_D\phi(w)$ for any $w \in D$. On the other hand, $f_n(w_0) \ge u_n(w_0) \to H_D\phi(w_0)$. Hence $f_n(w_0) \to H_D\phi(w_0)$.

The importance of the Perron function for the Dirichlet problem is in the following. If the Dirichlet problem has a solution then it is given by the Perron function. Indeed, if h is harmonic and $\lim_{z\to\zeta} h(z) = \phi(\zeta)$ on ∂D, then $h \in \mathcal{U}$. Therefore, $h \leq H_D\phi$. On the other hand if $u \in \mathcal{U}$ then $\limsup_{z\to\zeta}(u-h)(z) \leq 0$ for $\zeta \in \partial D$, and by the maximum principle $u \leq h$ on D, which implies that $H_D\phi \leq h$. Hence $H_D\phi = h$.

However, the existence of a solution is not a foregone conclusion as the following example shows. Consider the Dirichlet problem corresponding to $D = \Delta(0,1)\backslash\{0\}$ (punctured disk) and the function ϕ given by $\phi(\zeta) = 0$ for $|\zeta| = 1$ and $\phi(0) = -1$. If $h \in \mathcal{U}$ then by the maximum principle $h \leq 0$ on D, so $H_D\phi \leq 0$. Note that $\epsilon \ln|z| \in \mathcal{U}$ for every $\epsilon > 0$, by letting $\epsilon \to 0$ we conclude that $H_D\phi = 0$, which obviously doesn't solve the Dirichlet problem.

Definition Let D be a bounded domain in $\mathbb{C}$. A point $\zeta \in \partial D$ is called regular if $\lim_{z\to\zeta} H_D\phi(z) = \phi(\zeta)$ for every continuous $\phi : \partial D \to \mathbb{R}$. Domains whose boundaries consist of regular points only are called regular (so that the solution to the Dirichlet problem exists for continuous boundary conditions).

The class of regular domains is broad. We don't have time to go through derivations and only state a key result.

A simply connected domain bounded by a finite number of smooth boundary curves and such that its boundary is a simple closed curve is called a Jordan domain.

Thm 51. *Every Jordan domain is regular.*

More generally:

Thm 52. *If D is a simply connected domain such that $\mathbb{C}_\infty\backslash D$ contains at least two points then D is a regular domain; here $\mathbb{C}_\infty$ is the Riemann sphere.*

And

Thm 53. *Let D be a subdomain of $\mathbb{C}_\infty$, $\zeta_0 \in \partial D$, and C be the component of ∂D containing ζ_0. If $C \neq \{\zeta_0\}$ then ζ_0 is regular.*

The set of irregular points is a polar set. This statement is known as Kellogg's theorem.

(Lecture 5, 26 March 2012)

4.2 Harmonic Measure

The Perron function is a good theoretical concept but it does not give a clear recipe for solving the Dirichlet problem. Today we will briefly survey another approach which is based on a generalization of the Poisson integral and conformal mappings.

Definition (Harmonic Measure) Let D be a proper subdomain of $\mathbb{C}$ with a regular boundary (i.e. the Dirichlet problem has a solution), and let $\mathcal{B}(\partial D)$ be the Borel σ-algebra on ∂D. A harmonic measure for D is a function $\omega_D : D \times \mathcal{B}(\partial D) \to [0, 1]$, such that for every $z \in D$, $\omega_D(z, \cdot)$ is a probability measure on $\mathcal{B}(\partial D)$ and such that if $\phi : \partial D \to \mathbb{R}$ is a bounded continuous function on the boundary of D then the Perron function $H_D\phi$ (i.e. solution of the Dirichlet problem with boundary condition ϕ) is the generalized Poisson integral

$$H_D\phi(z) = \int_{\partial D} \phi(\zeta) d\omega_D(z, \zeta) \ .$$

It can be shown, see e.g. Ransford's book, that if D is a proper subdomain of $\mathbb{C}$ whose boundary is regular then there exists a unique harmonic measure for D.

Let Δ be the unit disk $\Delta(0, 1)$ and $\phi(e^{it})$ is continuous. Then the Poisson integral theorem asserts that the function

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} \phi(e^{it}) \frac{1 - |z|^2}{|e^{it} - z|^2} \, dt$$

is harmonic in Δ and $\lim_{z\to\zeta} h(z) = \phi(\zeta)$ for $|zeta| = 1$, so that it solves the associated Dirichlet problem. In other words, $h = H_\Delta\phi$. Hence, by inspection,

$$d\omega_\Delta(z, \zeta) = \frac{1}{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} \, dt$$

is the harmonic measure for the unit disk.

Recall that a domain D is called Jordan if it is simply connected and is bounded by a finite number of smooth curves such that ∂D is a simple closed curve. Every Jordan domain is regular, so its harmonic measure is unique. It turns out that the harmonic measure for a Jordan domain D can be found by making use of a conformal map $f : D \to \Delta$ and the Poisson integral.

The following theorem provides the necessary technical statement.

Thm 54. *(Caratheodory's Theorem) Let D be a Jordan domain and let $f : \Delta \to D$ be a conformal map from the unit disk Δ to D. Then f has a continuous extension to the boundary of Δ, and this extension is one-to-one from $\bar{\Delta}$ to $\bar{D}$.*

The existence of a conformal map form Δ to D is guaranteed by the Riemann mapping theorem. Its continuous extension to the boundary is useful, of course, in the context of the Dirichlet problem.

Thm 55. *If D is a Jordan domain and $\phi : \partial D \to \mathbb{R}$ is continuous and bounded, and, also, f is a conformal map from D to Δ, extended by Caratheodory's theorem, then the function $g(z) = h(f(z))$ where*

$$h(w) = \frac{1}{2\pi} \int_0^{2\pi} \phi(f^{-1}(e^{it})) \, \frac{1 - |w|^2}{|e^{it} - w|^2} \, dt$$

solves the Dirichlet problem on D with boundary condition ϕ.

Proof. The function $\phi(f^{-1}(e^{it}))$ is continuous, so by the Poisson integral theorem h is harmonic on Δ and $h(e^{it}) = \phi(f^{-1}(e^{it}))$. Consequently, $h(f(z)) = \phi(z)$ for every $z \in \partial D$. So it remains to prove that $g(z) = h(f(z))$ is harmonic on D.

Since Δ is simply connected, there exists a holomorphic function F on Δ such that $h = \operatorname{Re} F$, and $g = h \circ f = \operatorname{Re} F \circ f$. Obviously $F \circ f$ is holomorphic, so its real part is harmonic. Thus g is harmonic. □

The theorem above has an immediate consequence for determining the harmonic measure for Jordan domains.

Thm 56. *(Conformal Invariance of Harmonic Measure) Let D be a Jordan domain, f be a conformal map from D onto the unit disk Δ (extended by Caratheodory's theorem), and B is a Borel measurable set of ∂D. Then*

$$\omega_D(z, B) = \omega_\Delta(f(z), f(B)) = \int_B \frac{1 - |f(z)|^2}{|e^{it} - f(z)|^2} \frac{dt}{2\pi} \, .$$

Example. Let D be the upper half of the complex plane, and $f(z) = \frac{z-i}{z+i}$, then f a conformal map of D onto Δ. By conformal invariance

$$\omega_D(z, B) = \omega_\Delta(f(z), f(B)) = \int_B \frac{1 - |f(z)|^2}{|f(s) - f(z)|^2} |f'(s)| \frac{ds}{2\pi} = \int_B \frac{\operatorname{Im} z}{|z - s|^2} \frac{ds}{\pi}.$$

With $z = x + iy$,

$$\omega_D(x + iy, B) = \int_B \frac{y}{(x - s)^2 + y^2} \frac{ds}{\pi} \, .$$

This is the Poisson kernel for the upper half of the complex plane.

5 Capacity and Transfinite Diameter

Recall the concept of equilibrium measure. For any finite Borel measure μ with compact support, its energy $I(\mu)$ is

$$I(\mu) = \int\int \ln|z-w|\, d\mu(z)\, d\mu(w)\ .$$

We have proved that if E is compact then there exist at least one Borel probability measure μ_E such that

$$I(\mu_E) \geq I(\mu) \quad \text{for any other Borel probability measure } \mu \text{ on } E.$$

If, in addition, E is non-polar then this measure is unique.

Definition (Logarithmic Capacity) Let E be a compact subset of $\mathbb{C}$ and μ_E be its equilibrium measure. The logarithmic capacity of E is defined by

$$\text{cap}(E) = e^{I(\mu_E)}\ .$$

If E is polar then $I(\mu_E) = -\infty$ for every equilibrium measure, so that $\text{cap}(E) = 0$ for polar sets (polar sets hold no charge). Equally, if $\text{cap}(E) = 0$ then E is polar.

For non-compact sets E capacity is defined by as the supremum of $\text{cap}(K)$ over all compact subsets K of E. By definition $0 \leq \text{cap}(E) \leq +\infty$, however compact sets have finite capacity. It is a straightforward consequence of definition that $\text{cap}(E_1) \leq \text{cap}(E_2)$ if $E_1 \subset E_2$.

Somewhat surprisingly, capacity is related to geometry.

Definition (n-th Diameter) Let E be a compact set in $\mathbb{C}$. Then its n-th diameter is given by

$$\delta_n(E) = \sup_{z_1,\dots,z_n\in E} \left(\prod_{1\leq j<k\leq n} |z_k - z_j|\right)^{\frac{2}{n(n-1)}}\ , \quad (n \geq 2) \tag{5.1}$$

where the supremum is taken over all n-tuples of E.

Since E is compact, the supremum in (5.1) is always attained on some n-tuple. Any such n-tuple $(z_1^{(n)},\dots,z_n^{(n)})$ is called an n-point *Fekete set* for E, and the points $z_j^{(n)}$ are called *Fekete points.* The maximization in (5.1) has geometric meaning – we want to place n points on E in such a way that the geometric mean of the pairwise distances was the greatest. There are $n(n-1)$ distinct pairs among n points, hence the power in (5.1).

If $n = 2$ then the maximization problem in (4.1) gives the diameter of E, $\delta_2(E) = \max_{z_1,z_2 \in E} |z_1 - z_2|$. Obviously, any Fekete points lie on the boundary of E. This is also true in the general case $n \geq 2$, all Fekete points lie on the outer boundary $\partial_e E$ of E, that is, the boundary of the unbounded component of the complement of E (follows from the maximum modulus principle for holomorphic functions?).

The product in (5.1) is related to the Vandermonde determinant

$$\det(z_j^{k-1})_{j,k=1}^n = \prod_{1 \leq j < k \leq n} (z_k - z_j) , \tag{5.2}$$

hence an n-point Fekete set maximizes the modulus of the Vandermonde determinant over all n-tuples of E. This observation helps to compute the n-diameter of a disk.

Lemma 57. *Let $\bar{\Delta}$ be the closed unit disk about the origin. Then the set of n-th roots of unity is a Fekete set for $\bar{\Delta}$ and $\delta_n(\bar{\Delta}) = n^{1/(n-1)}$.*

Proof. It follows from Hadamard's inequality for determinants $|\det(b_{jk})| \leq \prod_j \left(\sum_k |b_{jk}|^2 \right)^{1/2}$ and relation (5.2) that

$$\sup_{z_1,\dots,z_n \in \bar{\Delta}} \prod_{1 \leq j < k \leq n} |z_k - z_j| \leq n^{n/2} .$$

We shall now show that the supremum is attained on n-th roots of unity $z_j^{(n)} = e^{i2\pi j/n}$, $j = 1, \dots, n$. To this end, note that

$$|\det(z_j^{k-1})_{j,k=1}^n|^2 = \det((z_j^{k-1}))^T \det(\bar{z}_j^{k-1}) = \det \Big(\sum_{l=1}^n z_j^{l-1} \bar{z}_k^{l-1} \Big)_{j,k=1}^n , \tag{5.3}$$

where $(...)^T$ stands for matrix transpose. Since powers of the roots of unity are orthogonal, $\sum_{l=0}^{n-1} (e^{i2\pi j/n})^l (e^{-i2\pi k/n})^l = n\delta_{j,k}$, the determinant on the right in (5.3) is diagonal and easy to compute, leading to

$$\prod_{1 \leq j < k \leq n} |e^{i2\pi k/n} - e^{i2\pi j/n}| = |\det((e^{i2\pi j/n})^{k-1})| = n^{n/2} .$$

Hence the n-th roots of unity is a Fekete set, and $\delta_n(\bar{\Delta}) = n^{1/(n-1)}$. □

On taking the log , it is apparent that the maximization problem in (5.1) one obtains an equivalent maximization problem

$$\mathcal{E}_n(E) = \sup_{z_1,\dots,z_n \in E} \sum_{1 \leq j < k \leq n} \ln |z_k - z_i| . \tag{5.4}$$

Obviously,

$$\mathcal{E}_n(E) = \frac{n(n-1)}{2} \ln \delta_n(E) \ . \tag{5.5}$$

Changing the sign in front of the logarithm, the optimization problem in (5.4) can be interpreted as finding the configuration of n equal charges confined to E which has minimal energy. In this context, a Fekete set represents an equilibrium configuration (as the energy of such configuration is minimal), and it is a natural question to ask about the distribution of the equilibrium configuration in the limit of large number of charges, $n \to \infty$.

Lemma 58. *(Transfinite Diameter) The sequence $\delta_n(E)$ is decreasing (non-increasing), i.e. $\delta_1(E) \geq \delta_2(E) \geq \delta_3(E) \ldots$, and, hence, has a limit,*

$$\tau(E) = \lim_{n\to\infty} \delta_n(E) \ .$$

This limit is called the transfinite diameter of E.

Proof. It is instructive to consider first $\delta_2(E)$ and $\delta_3(E)$, or equivalently, $\mathcal{E}_2(E)$ and $\mathcal{E}_3(E)$. If $(z_1^{(3)}, z_2^{(3)}, z_3^{(3)})$ is a Fekete 3-tuple, then

$$\mathcal{E}_3(E) = \ln|z_3^{(3)} - z_1^{(3)}| + \ln|z_3^{(3)} - z_2^{(3)}| + \ln|z_2^{(3)} - z_1^{(3)}|$$

and, as $\ln|z_3^{(3)} - z_2^{(3)}| \leq \mathcal{E}_2(E)$,

$$\mathcal{E}_3(E) \leq \ln|z_3^{(3)} - z_1^{(3)}| + \ln|z_2^{(3)} - z_1^{(3)}| + \mathcal{E}_2(E) \ .$$

Similarly,

$$\mathcal{E}_3(E) \leq \ln|z_3^{(3)} - z_2^{(3)}| + \ln|z_2^{(3)} - z_1^{(3)}| + \mathcal{E}_2(E) \ ,$$

and

$$\mathcal{E}_3(E) \leq \ln|z_3^{(3)} - z_2^{(3)}| + \ln|z_3^{(3)} - z_1^{(3)}| + \mathcal{E}_2(E) \ ,$$

Adding the three inequalities together, $3\mathcal{E}_3(E) \leq 2\mathcal{E}_3(E) + 3\mathcal{E}_2(E)$. Hence $\mathcal{E}_3(E) \leq 3\mathcal{E}_2(E)$, and $\ln \delta_3(E) \leq \ln \delta_2(E)$, in view of (5.5)

The general case of arbitrary n can be tackled along similar lines. Let $(z_1^{(n)}, z_2^{(n)}, \ldots, z_n^{(n)})$ is a Fekete n-tuple. For each $k = 1, \ldots, n$

$$\mathcal{E}_n(E) \leq \sum_{j, j\neq k} \ln|z_k^{(n)} - z_j^{(n)}| + \mathcal{E}_{n-1}(E) \ .$$

Adding these n inequalities together, $(n-2)\mathcal{E}_n(E) \leq n\mathcal{E}_{n-1}(E)$, so dividing through by $n(n-1)(n-2)$ we get the result. □

For the closed unit disk, $\delta_n(\bar{\Delta}) = n^{1/(n-1)}$. By letting $n \to \infty$ one concludes that the transfinite diameter of the closed unit disk is $\tau(\bar{\Delta}) = 1$. By examining the proof of Lemma 57, it is apparent that the roots of unity are also Fekete points for the unit circle. Thus $\delta_n(\partial\bar{\Delta}) = n^{1/(n-1)}$ and $\tau(\partial\bar{\Delta}) = 1$.

Let μ_n be the normalised counting measure of the n-th roots of unity. Its log-potential is $p_n(z) = \frac{1}{n}\sum_{j=1}^{n} \ln|z - e^{i2\pi j/n}|$. We recognise this as an integral sum. Thus, in the limit $n \to \infty$, for $|z| \neq 1$,

$$p_n(z) \to \frac{1}{2\pi}\int_0^{2\pi} \ln|z - e^{it}|\, dt = \begin{cases} \frac{1}{2\pi}\ln|z|, & \text{if } |z| > 1, \\ 0, & \text{if } |z| < 1. \end{cases}$$

By Widom's lemma, $\mu_n \to \frac{1}{2\pi}dt$ in the sense of weak convergence of measures, so that the limiting distribution of Fekete points (or the limiting equilibrium charge distribution) is uniform on the unit circle. By continuity $p(z) = 0$ on the unit circle, hence $\operatorname{cap}(\bar{\Delta}) = \tau(\bar{\Delta})$. This is part of a more general picture.

Thm 59. *(Fekete-Szegö Theorem) For any compact set E in $\mathbb{C}$,*

$$\operatorname{cap}(E) = \tau(E)\ .$$

Moreover, if E has positive capacity, then in the limit $n \to \infty$ the normalised counting measure $\mu_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{z_j^{(n)}}$ of a Fekete set $(z_1^{(n)}, z_2^{(n)}, \ldots, z_n^{(n)})$ converges weakly to the equilibrium measure μ_E of E.

Proof. Consider the function $F(z_1, \ldots, z_n) = \sum_{1\leq j<k\leq n} \ln|z_k - z_j|$. The equilibrium measure has unit mass and $F(z_1, \ldots, z_n) \leq \mathcal{E}_n(E)$, hence

$$\int \cdots \int F(z_1, \ldots, z_n) d\mu_E(z_1) \cdots d\mu_E(z_n) \leq \mathcal{E}_n(E) = \frac{n(n-1)}{2} \ln \delta_n(E)\ .$$

But

$$\int \cdots \int F(z_1, \ldots, z_n) d\mu_E(z_1) \cdots d\mu_E(z_n) = \frac{n(n-1)}{2} I(\mu_E)\ .$$

Therefore

$$\operatorname{cap}(E) \equiv e^{I(\mu_E)} \leq \tau(E)\ . \tag{5.6}$$

On the other hand, by the Helly-Bray selection principle, we can always find a weakly converging subsequence of $(\mu_n)_n$ (as these measures have unit mass and supported on a compact set). To simplify the notation, let $\mu_n \to \tilde{\mu}$ as $n \to \infty$. On regularizing the log-function

$$f_\epsilon(z) = \begin{cases} \ln|z|, & \text{if } |z| \geq \epsilon, \\ \ln\epsilon, & \text{if } |z| < \epsilon, \end{cases}$$

we have, by the monotone convergence theorem,

$$I(\tilde{\mu}) = \int\int \ln|z-w| d\tilde{\mu}(z) d\tilde{\mu}(w) = \lim_{\epsilon\downarrow 0} \int\int f_\epsilon(z-w) d\tilde{\mu}(z) d\tilde{\mu}(w).$$

Recalling the Stone-Weierstrass approximation theorem, by the weak convergence of measures,

$$\int\int f_\epsilon(z-w) d\tilde{\mu}(z) d\tilde{\mu}(w) = \lim_{n\to\infty} \int\int f_\epsilon(z-w) d\mu_n(z) d\mu_n(w)$$

The right-hand side can be written in terms of the Fekete n-tuples,

$$\begin{aligned}
\int\int f_\epsilon(z-w) d\mu_n(z) &= \frac{1}{n^2} \sum_{j,k=1}^{n} f_\epsilon(z_j^{(n)} - z_k^{(n)}) \\
&= \frac{2}{n^2} \sum_{1\le j<k\le n} f_\epsilon(z_j^{(n)} - z_k^{(n)}) + \frac{n}{n^2} \ln\epsilon \\
&\ge \frac{2}{n^2} \mathcal{E}_n(E) + \frac{1}{n} \ln\epsilon.
\end{aligned}$$

Therefore, for any $\epsilon > 0$,

$$\int\int f_\epsilon(z-w) d\tilde{\mu}(z) d\tilde{\mu}(w) \ge \lim_{n\to\infty} \Big(\frac{2}{n^2} \mathcal{E}_n(E) + \frac{1}{n} \ln\epsilon \Big) = \ln\tau(E) \ ,$$

and, hence, $I(\tilde{\mu}) \ge \ln\tau(E)$. Since $\ln\tau(E) \ge I(\mu_E)$, see (5.6), one concludes that $I(\tilde{\mu}) \ge I(\mu_E)$, hence, by definition of equilibrium measure, $\tilde{\mu} = \mu_E$.

Thus we have proved that $I(\tilde{\mu}) \ge \ln\tau(E)$ and $I(\tilde{\mu}) \le \ln\tau(E)$. Hence one conclude that $I(\tilde{\mu}) = \ln\tau(E)$, and $\mathrm{cap}(E) = \tau(E)$. □

An immediate consequence of this theorem, the uniqueness of the equilibrium measure, and our calculations of the limiting counting measure of the n-th roots of unity is the conclusion that the equilibrium measure of the closed unit disk is nothing else as the Lebesgue measure on the unit circle normalised to have arc-length one.