LTCC: Stochastic Processes (2)

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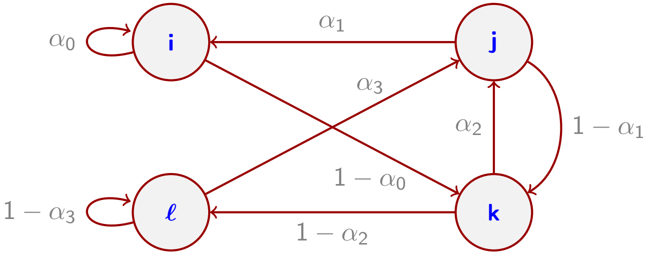


Homework Solution

Define the state space of Y as $\{0,1\}$ such that:

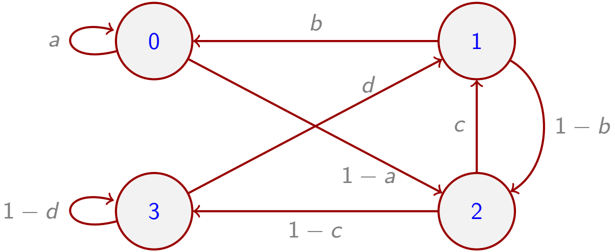
$$Y_n := \begin{cases} 1, & \text{sunny on } n\text{th day} \\ 0, & \text{rains on } n\text{th day} \end{cases}$$

Define $X_n := Y_{n-1} + 2Y_n$. Then we have state space of X is $\{0,1,2,3\}$:

X_n	Y_{n-1}	Y_n
0	0	0
1	1	0
2	0	1
3	1	1

X_n	Y_{n-1}	Y_n	$\mathbb{P}(Y_{n+1}=0 \mid Y_n, Y_{n-1})$	$\mathbb{P}(Y_{n+1}=1 \mid Y_n, Y_{n-1})$
0	0	0	$a = \mathbb{P}(X_{n+1}=0 \mid X_n=0) = p_{00}$	$1-a = \mathbb{P}(X_{n+1}=2 \mid X_n=0) = p_{02}$
1	1	0	$b = \mathbb{P}(X_{n+1}=0 \mid X_n=1) = p_{10}$	$1-b = \mathbb{P}(X_{n+1}=2 \mid X_n=1) = p_{12}$
2	0	1	$c = \mathbb{P}(X_{n+1}=1 \mid X_n=2) = p_{21}$	$1-c = \mathbb{P}(X_{n+1}=3 \mid X_n=2) = p_{23}$
3	1	1	$d = \mathbb{P}(X_{n+1}=1 \mid X_n=3) = p_{31}$	$1-d = \mathbb{P}(X_{n+1}=3 \mid X_n=3) = p_{33}$

state transition diagram



transition matrix

$$\mathbf{P} = \begin{bmatrix} a & 0 & 1-a & 0 \\ b & 0 & 1-b & 0 \\ 0 & c & 0 & 1-c \\ 0 & d & 0 & 1-d \end{bmatrix}$$

Outline

1. 1st step arguments
   - duration in state
   - jump probabilities
   - absorption times

2. example
   - homogenous difference equations
   - inhomogenous difference equations

3. classification of states
   - recurrence and transience
   - hitting times and return times
   - results

Remark 1

Recall, lecture 1, Markov processes are one-step away from independence. I.e. if X is discrete time Markov chain, then

$$\begin{aligned}\mathbb{P}\big(X_0 = i_0, \ldots, X_{n-1} = i_{n-1}\big) &= \mathbb{P}\big(X_0 = i_0\big) \prod_{k=1}^{n-1} \mathbb{P}\big(X_k = i_k \big| X_{k-1} = i_{k-1}\big) \\ &= \mathbb{P}\big(X_0 = i_0\big)\, p_{i_0\, i_1}\, p_{i_1\, i_2} \cdots p_{i_{n-2}\, i_{n-1}}\end{aligned}$$

Proposition 2 (Time spent in a (Markov chain) state is geometric)

Proof7 Given Markov X starts in state i it moves to a different state at time n with probability

$$\begin{aligned}&\mathbb{P}\big(X_n \neq i, X_{n-1} = \cdots = X_1 = i \big| X_0 = i\big) \\ =\;& \mathbb{P}\big(X_n \neq i \big| X_{n-1} = \cdots = X_1 = X_0 = i\big)\, \mathbb{P}\big(X_{n-1} = \cdots = X_1 = i \big| X_0 = i\big) \\ =\;& \big(1 - \mathbb{P}\big(X_n = i \big| X_{n-1} = \cdots = X_1 = X_0 = i\big)\big)\, (p_{ii})^{n-1} \text{ [from Remark 1]} \\ =\;& (1 - p_{i\,i})\, p_{i\,i}^{n-1}\end{aligned}$$

$$\mathbb{P}(X_n \neq i, X_{n-1} = \cdots = X_1 = i \big| X_0 = i) = (1 - p_{i\,i})p_{i\,i}^{n-1}$$

But, this is also $\mathbb{P}(T = n)$, where $T =$ time spent in state i. I.e. r.v. T has geometric distribution $(1 - p_{i\,i})p_{i\,i}^{n-1}$ ■

Corollary 3

For any Markov chain, the average time spent in state i is:

$$\mathbb{E}(T) = \frac{1}{1 - p_{i\,i}}$$

Also: when a process leaves state i, it moves to state j with probability

$$\begin{aligned}
\mathbb{P}(X_n = j | X_n \neq i, X_{n-1} = i) &= \frac{\mathbb{P}(X_n = j \neq i, X_{n-1} = i)}{\mathbb{P}(X_n \neq i, X_{n-1} = i)} \\
&= \frac{\mathbb{P}(X_n = j \neq i \big| X_{n-1} = i)}{\mathbb{P}(X_n \neq i \big| X_{n-1} = i)} \frac{\mathbb{P}(X_{n-1} = i)}{\mathbb{P}(X_{n-1} = i)} \\
&= \frac{p_{i\,j}}{(1 - \mathbb{P}(X_n = i \big| X_{n-1} = i))}, \quad j \neq i \\
&= \frac{p_{i\,j}}{1 - p_{i\,i}}, \quad j \neq i
\end{aligned}$$

Definition 4 (absorbing state)

(Recall) i is an absorbing state iff $p_{ii} = 1$.

Let $\mathcal{C} \subset \mathcal{S}$ be the set of all absorbing states of a Markov chain X. Then for some $j \in \mathcal{S}$, define [prob. ever visiting j from i]

$$
\begin{aligned}
\theta_i &:= \mathbb{P}\big(X_n = j, n \geq 0 \big| X_0 = i\big) \\
&= \sum_{k \in \mathcal{S}} \mathbb{P}\big(X_1 = k \big| X_0 = i\big)\, \mathbb{P}\big(X_n = j, n \geq 0 \big| X_1 = k, X_0 = i\big) \\
&= \sum_{k \in \mathcal{S}} p_{ik} \mathbb{P}\big(X_n = j, n \geq 0 \big| X_1 = k\big) \\
\theta_i &= \sum_{k \in \mathcal{S}} p_{ik}\theta_k = p_{ii}\theta_i + \sum_{k \in \mathcal{S}\setminus\{i\}} p_{ik}\theta_k
\end{aligned}
$$

Then, for $i \notin \mathcal{C}$, solve for θ_i:

$$
\theta_i = (1 - p_{ii})^{-1} \sum_{k \in \mathcal{S}\setminus\{i\}} p_{ik}\theta_k
$$

Let $\mathcal{C} \subset \mathcal{S}$ be the set of all absorbing states of a Markov chain X. Then for some $i \in \mathcal{S}$, define [time to absorption from state i]

$$T_i := \min \left\{ n \geq 0 \colon X_n \in \mathcal{C} \middle| X_0 = i \right\}$$

and define the mean time to absorption $E_i := \mathbb{E} T_i$.

Then if $i \in \mathcal{C}$ $\mathbb{E} T_i = 0$ as $T_i = 0$.

If $i \notin \mathcal{C}$

$$\begin{aligned}
E_i &= \mathbb{E}\,\mathbb{E}\big(T_i|X_1\big) && \text{[total law expectation]} \\
&= \sum_{j\in\mathcal{S}} \mathbb{E}\big(T_i\big|X_1 = j\big)\,p_{ij} && \text{[still condition to } X_0 = i\text{]} \\
&\quad \text{move from } X_0 = i \text{ to } X_1 = j \\
&= \sum_{j\in\mathcal{S}} \big(1 + \mathbb{E}\,T_j\big)\,p_{ij} && [\{T_i|X_1 = j\} = 1 + T_j] \\
&\quad \text{if } j \in \mathcal{C},\ \ T_i = 1;\ \text{if not we need to move from } j \\
&= \sum_{j\in\mathcal{S}} \big(1 + E_j\big)\,p_{ij} \\
&= 1 + \sum_{j\in\mathcal{S}} E_j p_{ij} = 1 + \sum_{j\in\mathcal{S}\backslash\mathcal{C}} E_j p_{ij} \\
&\quad \text{because } E_j = 0 \text{ if } j \in \mathcal{C}
\end{aligned}$$

Etc.

Similarly,

$$\begin{aligned}
\mathbb{E}\,\mathbb{E}(T_i^2|X_1) &= \sum_{j\in\mathcal{S}} \mathbb{E}(T_i^2 \big| X_1 = j)\, p_{ij} \\
&= \sum_{j\in\mathcal{S}} \mathbb{E}(1+T_j)^2\, p_{ij} \qquad [\{T_i^2|X_1=j\} = (1+T_j)^2] \\
&= \sum_{j\in\mathcal{S}} \left(1 + 2E_j + \mathbb{E}\,T_j^2\right) p_{ij} \\
&= 1 + \sum_{j\in\mathcal{S}} \left(2E_j + \mathbb{E}\,T_j^2\right) p_{ij} \\
&= 1 + \sum_{j\in\mathcal{S}\setminus\mathcal{C}} \left(2E_j + \mathbb{E}\,T_j^2\right) p_{ij}
\end{aligned}$$

I.e., find E_j first and now solve for $\mathbb{E}\,T_j^2$ to find var T_j.

Remark 5

Very useful to calculate certain properties of a Markov chain. See following two part example. 1st exploits law of total prob.; 2nd uses law of total expectation. Both yield 'simple' difference equations.

Example 6 (gambler's ruin: prob. of absorption)

Recall $X_n = \#\text{chips } A$ *has after* n *games.*

$$\begin{aligned} \mathbb{P}(A \text{ wins}) &= p \\ \mathbb{P}(B \text{ wins}) &= 1 - p =: q \end{aligned}$$

Probability of 'absorption' in state $a + b$, *given chain starts at* i *is*

$$\theta_i := \mathbb{P}\big(X_n = a + b, \text{for some } n \geq 0 \big| X_0 = i\big)$$

Find θ_i *in terms of* p, q, a, b.

Solution (part I) [Law of total prob.:] $\theta_i = \mathbb{P}\big(X_n = a+b,\ n \geq 0 \big| X_0 = i\big)$

$$= \sum_{k:\ i+k \in \mathcal{S}} \mathbb{P}\big(X_1 = i+k \big| X_0 = i\big)\, \mathbb{P}\big(X_n = a+b,\ n \geq 0 \big| X_0 = i, X_1 = i+k\big)$$

[Markov ppty:]

$$\begin{aligned}
&= \sum_{k:\ i+k \in \mathcal{S}} \mathbb{P}\big(X_1 = i+k \big| X_0 = i\big)\, \mathbb{P}\big(X_n = a+b,\ n \geq 0 \big| X_1 = i+k\big) \\
&= \sum_{k=\pm 1} \mathbb{P}\big(X_1 = i+k \big| X_0 = i\big)\, \mathbb{P}\big(X_n = a+b,\ n \geq 0 \big| X_1 = i+k\big) \\
&= p\, \mathbb{P}\big(X_n = a+b,\ n \geq 0 \big| X_1 = i+1\big) \\
&\quad + q\ \mathbb{P}\big(X_n = a+b,\ n \geq 0 \big| X_1 = i-1\big) \\
&= p\, \mathbb{P}\big(X_{n-1} = a+b,\ n \geq 1 \big| X_0 = i+1\big) \\
&\quad + q\ \mathbb{P}\big(X_{n-1} = a+b,\ n \geq 1 \big| X_0 = i-1\big) \quad \text{[time-homog]}
\end{aligned}$$

and we have the difference equation

$$\theta_i = p\,\theta_{i+1} + q\,\theta_{i-1}\,, \quad i = 1, \ldots, a+b-1 \tag{1}$$

with boundary conditions $\theta_0 = 0$, and $\theta_{a+b} = 1$.

Homogenous 2nd order difference equation:

$$\theta_i = p\,\theta_{i+1} + q\,\theta_{i-1}\,, \quad i = 1, \ldots, a+b-1 \tag{2}$$

with boundary conditions $\theta_0 = 0$, and $\theta_{a+b} = 1$.
Following the usual (difference equation approach, we 'try' a solution of the form $\theta_i = w^i$.

$$\begin{aligned} w^i &= pw^{i+1} + qw^{i-1} \\ w &= pw^2 + q \\ pw^2 - w + q &= 0 \text{ solution of quadratic form} \\ w &= \frac{1 \pm \sqrt{1-4pq}}{2p} = \frac{1 \pm \sqrt{1-4p(1-p)}}{2p} \\ &= \frac{1 \pm (1-2p)}{2p} = \begin{cases} 1 \\ q/p \end{cases} \end{aligned}$$

There are now (unfortunately) 2 cases that must be considered: distinct roots ($\rho = q/p \neq 1$) and repeated roots ($\rho = 1$).

Case 1: $\rho \neq 1$.

General solution

$$\theta_i = A_1 1^i + A_2 \left(\frac{q}{p}\right)^i$$

from boundary condition

$$\begin{aligned}
1 &= A_1 + A_2 \left(\frac{q}{p}\right)^{a+b} \\
0 &= A_1 + A_2 \\
\Rightarrow 1 &= A_1 - A_1 \left(\frac{q}{p}\right)^{a+b} = A_1 \left(1 - \left(\frac{q}{p}\right)^{a+b}\right) \\
A_1 &= \frac{1}{1-(q/p)^{a+b}}
\end{aligned}$$

$$\theta_i = \frac{1-(q/p)^i}{1-(q/p)^{a+b}} = \frac{1-\rho^2}{1-\rho^{a+b}}$$

Case 2: $\rho = 1.$

If $q/p = 1$ then $w_1 = w_2 = 1$ and the general solution is

$$\theta_i = A_1 + A_2 i$$

Now use boundary conditions to find constants A_1 and A_2. For

$$\begin{aligned} \theta_0 &= 0 = A_1 \\ \theta_{a+b} &= 1 = A_2(a+b) \Rightarrow A_2 = \frac{1}{a+b} \end{aligned}$$

We obtain

$$\theta_i = i/(a+b)$$

.

Example 7 (gambler's ruin: expected time of absorption)

Let $T_i :=$ *time to absorption (state* 0 *or* $a+b$*), given* $X_0 = i$*. I.e.*

$$T_i := \min\{n \geq 0 \colon X_n = 0 \text{ or } X_n = a+b \,|\, X_0 = i\}$$

Find expected absorption time in terms of p, q, a, b*.*

Solution (part I) [Law of total expectation:]

$$\begin{aligned}
\mathbb{E}(T_i) &= \mathbb{E}(\mathbb{E}(T_i | X_1 = j)) \\
&= \sum_{j \in \mathcal{S}} \mathbb{P}(X_1 = j | X_0 = i)\, \mathbb{E}(T_i | X_1 = j) \\
&= \sum_{j \in \mathcal{S}} p_{i\,j}\, \mathbb{E}(T_i | X_1 = j)
\end{aligned}$$

But

$$p_{ij} = \begin{cases} p, & \text{j=i+1} \\ q, & \text{j=i-1} \\ 0, & oth. \end{cases}$$

Hence

$$\mathbb{E}(T_i) = p\, \mathbb{E}(T_i | X_1 = i+1) + q\, \mathbb{E}(T_i | X_1 = i-1)$$

$$\mathbb{E}(T_i) = p\,\mathbb{E}(T_i|X_1 = i+1) + q\,\mathbb{E}(T_i|X_1 = i-1)$$

Now recall

$$\begin{aligned} \{T_i|X_1 = j\} &= \text{absorption time, given } X_1 = j \\ &= 1 + \text{absorption time, given } X_0 = j \\ &= 1 + T_j \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E}(T_i) &= p\,\mathbb{E}(1 + T_{i+1}) + q\,\mathbb{E}(1 + T_{i-1}) \\ &= p\,(1 + \mathbb{E}(T_{i+1})) + q\,(1 + \mathbb{E}(T_{i-1})) \\ &= 1 + p\,\mathbb{E}(T_{i+1}) + q\,\mathbb{E}(T_{i-1}) \qquad [p + q = 1] \end{aligned}$$

and we have another difference equation

$$E_i = 1 + p\,E_{i+1} + q\,E_{i-1}\,, \qquad i = 1, \ldots, a+b-1 \tag{3}$$

with boundary conditions $E_0 = 0$, and $E_{a+b} = 0$.

Now, recall Solution, part I of Example 7:

$$p\,E_{i+1} + q\,E_{i-1} - E_i = -1\,, \qquad i = 1, \ldots, a+b-1 \tag{4}$$

with boundary conditions $E_0 = 0$, and $E_{a+b} = 0$.

Remark 8

Note 1st difference equation (Eq. 1) was homogenous: of the form $\sum_k a_k y_{i-k} = 0$. The above equation is inhomogenous: of the form $\sum_k a_k y_{i-k} = x_i$. Therefore the general solution takes the form

$$E_i = \underbrace{E_i^{(c)}}_{\text{complementary solution}} + \underbrace{E_i^{(p)}}_{\text{particular solution}}$$

For the complementary solution, we solve

$$p\,E_{i+1} + q\,E_{i-1} - E_i = 0$$

But, this is same form as 1st difference equation (1). Hence complementary solution unfortunately has 2 cases and the first case ($\rho \neq 1$) takes form

$$E_i^{(c)} = c_1 + c_2\rho^i \tag{5}$$

For the particular solution, we use 'lucky guess method' (!): this solution has to be such that when substitute in $p\,E_{i+1} + q\,E_{i-1} - E_i$ the result is -1
Try putting

$$E_i^{(p)} = \alpha i + \beta$$

into Eq. 4:

$$\begin{aligned} p\big(\alpha(i+1)+\beta\big) + q\big(\alpha(i-1)+\beta\big) - \alpha i - \beta &= -1 \\ &\Downarrow \\ p\alpha - q\alpha &= -1 \end{aligned}$$

Comparing constant (i^0) terms gives:

$$\alpha = \frac{1}{q-p}\,, \text{ i.e. } \quad E_i^{(p)} = \frac{i}{q-p}$$

Case 1: $\rho \neq 1$.

General solution

$$E_i = E_i^{(c)} + E_i^{(p)} = c_1 + c_2\rho^i + \frac{i}{q-p}$$

Using boundary conditions:

$$i = 0: \; E_0 = 0 = c_1 + c_2 \Rightarrow c_2 = -c_1$$

and

$$i = a+b: \; E_{a+b} = 0 = c_1(1-\rho^{a+b}) + \frac{a+b}{q-p}$$

$$\Rightarrow c_1 = -\frac{a+b}{q-p} \quad \frac{1}{1-\rho^{a+b}}$$

which gives

$$E_i = \frac{1}{q-p}\Big(i - \frac{(a+b)(1-\rho^i)}{1-\rho^{a+b}}\Big), \qquad \rho \neq 1$$

Case 2: $\rho = 1$, i.e. $(p = q = 1/2)$

.
Eq. (4) is

$$\frac{1}{2}E_{i+1} + \frac{1}{2}E_{i-1} - E_i = -1 \tag{6}$$

For the complementary solution we solve

$$\frac{1}{2}E_{i+1} + \frac{1}{2}E_{i-1} - E_i = 0$$

but this is same form as Eq (1) with $\rho = 1$. Hence

$$E_i^{(c)} = c_1 + c_2 i$$

For the particular solution, we 'try' putting $E_i^{(p)} = \alpha i^2$ into Eq (6):

$$\frac{1}{2}\left(\alpha(i+1)^2\right) + \frac{1}{2}\left(\alpha(i-1)^2\right) - \alpha i^2 = -1$$

Equating constant terms gives $\alpha = -1$ and we have $E_i^{(p)} = -i^2$. So, for $\rho = 1$ the general solution takes the form

$$E_i = E_i^{(c)} + E_i^{(p)} = c_1 + c_2 i - i^2$$

General solution for $\rho = 1$:

$$E_i = E_i^{(c)} + E_i^{(p)} = c_1 + c_2 i - i^2$$

Using boundary conditions:

$$i = 0 : E_0 = 0 = c_1$$

$$i = a + b : E_{a+b} = 0 = c_2(a+b) - (a+b)^2 \Rightarrow c_2 = a + b$$

Therefore

$$E_i = (a+b)i - i^2 = i(a+b-1)\,, \qquad \rho = 1 \qquad \blacksquare$$

Remark 9

Knowing whether or not a Markov chain ever returns to a particular state is an important property worth knowing about. In fact, the state space of a Markov chain can be partitioned (by an equivalence relation) into states that are guaranteed to be visited at some point and those that are not. In some sense, this reveals the underlying structure of the Markov chain in question.

Definition 10

State i is recurrent (aka persistent, G&S p220) if the probability of ever returning to state i is 1, i.e.

$$\mathbb{P}\left(X_n = i, \text{ for some } n \geq 1 \middle| X_0 = i\right) = 1$$

If this probability is < 1 then state i is called transient.

Example 11

Random walk. Consider a random walk starting at zero with $\mathbb{P}(X_{i+1} = i+1 \mid X_i = i) = p$ *and* $\mathbb{P}(X_{i+1} = i-1 \mid X_i = i) = 1 - p = q$

It is clear that we cannot return to zero after an odd number of steps so that $p_{00}^{(2n+1)} = 0$ *for all* n*. Any given sequence of length* $2n$ *from 0 to 0 occurs with probability* $p^n q^n$*, there being* n *steps down and* n *steps up. The number of ways in which we can do this is* $\binom{2n}{n}$*. Therefore*

$$p_{00}^{2n} = \binom{2n}{n} p^n q^n$$

and 0 is a transient state,

Find general transition probabilities for the random walk.

Let S_m be iid rv with

$$S_m = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } q \end{cases}$$

Then $X_n = X_0 + S_1 + \cdots , S_n, \qquad n = 0, 1, \ldots$

For $m \geq 1$ the rv $\frac{1}{2}(S_m + 1)$ has Bernoulli distribution with probability of success p and so $B_n = \frac{1}{2}(X_n + n)$ has Binomial distribution with parameters n and p. Hence

$$\Pr(X_n = k \mid X_0 = 0) = \Pr(B_n = \frac{1}{2}(n+k)) = \binom{n}{\frac{1}{2}(n+k)} p^{\frac{1}{2}(n+k)} q^{\frac{1}{2}(n-k)}$$

whenever k is such that $\frac{1}{2}(n+k)$ is an integer between 0 and n.

Of interest here is, given we know the transition probabilities, (how) can we deduce whether a given state is recurrent or transient?

Definition 12

Define $f_{ij}(n)$ as the probability that the 1st visit to j from i occurs at time $n \geq 1$. I.e.

$$f_{ij}(n) := \begin{cases} \mathbb{P}\left(X_n = j, X_{1:n-1} \neq j \middle| X_0 = i\right), & n \geq 1 \\ 0, & n = 0 \end{cases}$$

where $\{X_{1:n-1}\} := \{X_k, 1 \leq k \leq n-1\}$.

Definition 13

The probability of ever visiting j from i is:

$$f_{ij} := \sum_{n=1}^{\infty} f_{ij}(n)$$

I.e. f_{jj} is probability of ever returning to j.

Lemma 14

A state j is

$$\text{recurrent iff} \quad f_{jj} = 1$$

$$\text{transient iff} \quad f_{jj} < 1$$

'Proof' By definition.

Definition 15

Define time to 1st visit to state j (aka 1st hitting time or 1st passage time) is

$$T_j := \min\{n \geq 1 : X_n = j\}$$

Remark 16

By definition, we have

$$\mathbb{P}\big(T_j = n \big| X_0 = i\big) = f_{ij}(n)$$

In particular $\mathbb{P}\big(T_j = n \big| X_0 = j\big) = f_{jj}(n)$ is the probability that the 1st return to j occurs at nth time step.

Remark 17

So, the classification of a state, i.e. whether it is recurrent/transient can be determined by T_j *or* $f_{ij}(n)$. *The next theorem provides the opportunity to relate transition probabilities with class.*

Theorem 18

Let $p_{ij}^{\sim}$ *be the generating function of* $p_{ij}^{(n)}$ *and* $f_{ij}^{\sim}$ *be the generating function of* $f_{ij}(n)$. *I.e.*

$$p_{ij}^{\sim}(s) := \sum_{n=0}^{\infty} s^n p_{ij}^{(n)}, \quad f_{ij}^{\sim}(s) := \sum_{n=0}^{\infty} s^n f_{ij}(n), \quad |s| < 1$$

with the convention $p_{ij}^{(0)} = \delta_{ij}$ *and* $f_{ij}(0) = 0$ *Then*

$$p_{ij}^{\sim}(s) = \delta_{ij} + f_{ij}^{\sim}(s) p_{jj}^{\sim}(s)$$

Proof

$$p_{ij}^{(n)} = \mathbb{P}(X_n = j | X_0 = i)$$
$$= \sum_{k=1}^{n} \mathbb{P}(X_n = j, T_j = k | X_0 = i) \qquad \text{[Total prob]}$$
$$= \sum_{k=1}^{n} \mathbb{P}(X_n = j | T_j = k, X_0 = i)\, \mathbb{P}(T_j = k | X_0 = i)$$

Now, $\{T_j = k, X_0 = i\} = \{X_k = j, X_{1:k-1} \neq j, X_0 = i\}$. Hence

$$p_{ij}^{(n)} = \sum_{k=1}^{n} \mathbb{P}(X_n = j | X_k = j, X_{1:k-1} \neq j, X_0 = i)\, f_{ij}(k)$$
$$= \sum_{k=1}^{n} \mathbb{P}(X_n = j | X_k = j)\, f_{ij}(k) \qquad \text{[Markov]}$$
$$= \sum_{k=1}^{n} \mathbb{P}(X_{n-k} = j | X_0 = j)\, f_{ij}(k) \qquad \text{[Time-homog.]}$$
$$= \sum_{k=1}^{n} p_{jj}^{(n-k)} f_{ij}(k)$$

$$p_{ij}^{(n)} = \sum_{k=1}^{n} p_{jj}^{(n-k)} f_{ij}(k)$$

Now, multiply both side by s^n and sum over $n = 1, \ldots, \infty$. LHS:

$$\sum_{n=1}^{\infty} s^n p_{ij}^{(n)} = -p_{ij}^{(0)} + \sum_{n=0}^{\infty} s^n p_{ij}^{(n)} = -\delta_{ij} + p_{ij}^{\sim}(s)$$

RHS:$\sum_{n=1}^{\infty} \sum_{k=1}^{n} s^n p_{jj}^{(n-k)} f_{ij}(k)$. We can use identity

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n} b_n a_{n-k,k} = \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} b_{m+k} a_{mk}$$ [check: ex. for reader!]

to get that the RHS is:

$$\sum_{m=0}^{\infty} \sum_{k=1}^{\infty} s^{m+k} p_{jj}^{(m)} f_{ij}(k) = \sum_{m=0}^{\infty} s^m p_{jj}^{(m)} \sum_{k=1}^{\infty} s^k f_{ij}(k) = p_{jj}^{\sim}(s)\, f_{ij}^{\sim}(s)$$ ■

[Nb, $f_{ij}(0) = 0$].

Corollary 19

$$\sum_{n=0}^{\infty} p_{jj}^{(n)} \begin{cases} = \infty, & \textit{iff } j \textit{ recurrent} \\ < \infty, & \textit{iff } j \textit{ transient} \end{cases}$$

'Proof' From Theorem 18, we had $p_{jj}^{\sim}(s) = 1 + p_{jj}^{\sim}(s) f_{jj}^{\sim}(s)$, i.e.

$$p_{jj}^{\sim}(s) = \frac{1}{1 - f_{jj}^{\sim}(s)} \tag{7}$$

Now take limit as $s \to 1^-$ (from below: recall $|s| < 1$):

$$\lim_{s \to 1^-} p_{jj}^{\sim}(s) = \lim_{s \to 1^-} \sum_{n=0}^{\infty} s^n p_{jj}^{(n)} = \sum_{n=0}^{\infty} p_{jj}^{(n)}$$

Also note that

$$\lim_{s \to 1^-} f_{jj}^{\sim}(s) = \lim_{s \to 1^-} \sum_{n=0}^{\infty} s^n f_{jj}(n) = \sum_{n=0}^{\infty} f_{jj}(n) = f_{jj}$$

An we have (informally)

$$\sum_{n=0}^{\infty} p_{jj}^{(n)} = \frac{1}{1 - f_{jj}}$$

$$\sum_{n=0}^{\infty} p_{jj}^{(n)} = \frac{1}{1 - f_{jj}}$$

I.e., since $f_{jj} \leq 1$ (it is a prob.) this converges iff $f_{jj} < 1$ and diverges iff $f_{jj} = 1$. Recall from Lemma 14 that j is transient iff $f_{jj} < 1$ and recurrent iff $f_{jj} = 1$ ■

Corollary 20

If j is transient, then $p_{ij}^{(n)} \to 0$, as $n \to \infty, \forall\, i \in \mathcal{S}$.

Proof From Theorem 18

$$p_{ij}^{\sim}(s) = \delta_{ij} + f_{ij}^{\sim}(s) p_{jj}^{\sim}(s)$$

Take limit

$$\lim_{s \to 1^-} \sum_{n=0}^{\infty} s^n p_{ij}^{(n)} = \delta_{ij} + \lim_{s \to 1^-} \sum_{k=0}^{\infty} s^k p_{jj}^{(k)} \sum_{m=0}^{\infty} s^m f_{ij}(m)$$

$$\sum_{n=0}^{\infty} p_{ij}^{(n)} = \delta_{ij} + \sum_{k=0}^{\infty} p_{jj}^{(k)} \sum_{m=0}^{\infty} f_{ij}(m)$$

$$\begin{aligned}\sum_{n=0}^{\infty} p_{ij}^{(n)} &= \delta_{ij} + \sum_{k=0}^{\infty} p_{jj}^{(k)} \sum_{m=0}^{\infty} f_{ij}(m) \\ &= \delta_{ij} + f_{ij} \underbrace{\sum_{k=0}^{\infty} p_{jj}^{(k)}}_{[< \infty \Leftrightarrow j \text{ transient}]}\end{aligned}$$

$\Rightarrow \sum_{n=0}^{\infty} p_{ij}^{(n)}$ converges, i.e. must have

$$p_{ij}^{(n)} \to 0, \text{ as } n \to \infty \qquad \blacksquare$$

Recall:

$$\sum_{n=0}^{\infty} p_{jj}^{(n)} \begin{cases} = \infty, & \textit{iff j recurrent} \\ < \infty, & \textit{iff j transient} \end{cases}$$

f_{jj} is prob. ever returning to j. Then $1 - f_{jj}$ is prob never returning to j.

Define $N_j(n)$ as # transitions into j up to time n.

In particular, define $N_j(\infty)$ as total # transitions into j. Then

$$\{N_j(\infty) \big| X_0 = j\}$$

is the total number of returns to j.

Remark 21

For j *recurrent*

$$\mathbb{P}(N_j(\infty) = k \big| X_0 = j) = \begin{cases} 0, & k < \infty \\ 1, & k \to \infty \end{cases}$$

For j *transient* $\mathbb{P}(N_j(\infty) = k \big| X_0 = j)$ *is prob. that chain returns to* j *a total # of* k *times and then never returns. I.e.*

$$\mathbb{P}(N_j(\infty) = k \big| X_0 = j) = f_{jj}^k (1 - f_{jj})$$

I.e. in this case $N_j(\infty) \big| X_0 = j \sim$ *geometric.*
For j *transient* $N_j(\infty) \big| X_0 = j \sim$ *geometric. I.e.*

$$\mathbb{E}(N_j(\infty) \big| X_0 = j) = \frac{f_{jj}}{1 - f_{jj}} < \infty \qquad [j \text{ transient} \Rightarrow f_{jj} < 1 c.f. Lemma (14)]$$

Conversely, for j *recurrent*

$$\mathbb{E}(N_j(\infty) \big| X_0 = j) = \infty$$