LTCC–2024-25: Stochastic processes

Attempt 4 out of the 7 questions.

Please typeset your solutions. You can do the coding in R, Python, C++, or C64 Basic. Do not use fancy packages. For example, if you are simulating a Markov chain, do not use a Markov chain package.

Question 1

Let X be a irreducible and aperiodic Markov chain on a finite state space containing distinct states $s, t, u \in S$. Consider the average

$$A_n = \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}[X_{k+2} = u, X_{k+1} = t, X_k = s].$$

Using a renewal type argument, prove that A_n converges almost-surely and argue that limit must be $\pi_s p_{st} p_{tu}$, where π is the stationary distribution and P is the transition matrix of X. Hint: you may use the fact that the limit does not depend on the distribution of X_0.

Question 2

Illustrate the results of Question 1, via simulations, using the following transition matrix

$$P = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/8 & 1/4 & 5/8 \end{bmatrix},$$

with $s = 1, t = 2$ and $u = 3$.

Question 3

Consider the point process Γ on $[0, \infty)$ defined in the following way. For every $i \in \mathbb{N}$, place an independent Poisson point process on the interval $[i, i+1)$ of intensity λ_i. Prove that Γ has infinitely many points almost surely if and only if the expected number of points of Γ is infinite.

Question 4

You are given Γ, a Poisson point process of intensity 3 on the interval $[0, \pi]$. Consider the function $g(t) = \sin(t) + 2$ and the following thinning procedure to produce another point process Γ'. Suppose the points of Γ are given by $\{t_1, \ldots, t_n\}$. We consider independent $\mathrm{Bern}(p_i)$ random variables, $X_1, \ldots, X_n$, with

$$p_i = g(t_i)/3.$$

We set $\Gamma' \subseteq \Gamma$, where t_i is a point of Γ' if and only if $X_i = 1$.

Let N be the total number of points of Γ' in the interval $[0, \pi]$. Prove that

$$\mathbb{E}N = \int_0^\pi g(t)dt = 2\pi + 2.$$

Hint: if $g \equiv 2$, then this question is easy, why?

Question 5

Illustrate the answer in Question 4 by running simulations.

Question 6

Consider a branching process Z with $Z_0 = 0$ and offspring distribution $p_0 = 1/8$, $p_1 = 3/8$, $p_2 = 3/8$, and $p_3 = 1/8$. Compute the extinction probability.

Question 7

Simulate the branching process in Question 6, and use your simulation to estimate the extinction probability. Depending on your patience, you may have to implement some practical choices to get satisfying answers.