Theory of Linear Models Exam Question

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An experiment yielded the following data on a response variable Y and explanatory variables X_1 , X_2 and X_3 :

X_1	X_2	X_3	Y
-1	-1	-1	12.3
-1	-1	1	19.7
-1	1	-1	14.3
-1	1	1	13.9
1	-1	-1	4.6
1	-1	1	5.7
1	1	-1	2.2
1	1	1	8.1
0	0	0	19.4
0	0	0	1.5
0	0	0	1.1
0	0	0	19.7

We propose a second order response surface model (including a constant) with Y_i s uncorrelated and $\operatorname{Var}[Y_i] = \sigma^2$, $i = 1, 2, \ldots, 12$, for these data.

- 1. Express the proposed model in matrix form, i.e. write down the model matrix \mathbf{X} and the vector of parameters $\boldsymbol{\beta}$.
- 2. Find the rank of the matrix **X**.
- 3. Find a generalized inverse of $\mathbf{X}'\mathbf{X}$.
- 4. Find a least squares estimator $\hat{\boldsymbol{\beta}}$ of $\boldsymbol{\beta}$. (Hint: Use the generalized inverse of $\mathbf{X}'\mathbf{X}$.)

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- 5. Show that β_{33} is not estimable.
- 6. Show that $\theta = \beta_{11} + \beta_{22} + \beta_{33}$ is estimable. Find the form and observed value of its least squares estimator $\hat{\theta}$.
- 7. Find the variance of $\hat{\theta}$, in terms of σ^2 .
- 8. Write down the form of a $100(1 \alpha)\%$ confidence interval for θ . You are not required to calculate this interval.