LTCC: Time Series Analysis
Concise Solutions to the Mock Exam

Part I: R

1. Prices often evolve in a "multiplicative" way according to percentage changes. For example, imagine the investment of one pound in a bank, where the interest rate is 3% yearly. After one year, we have 1.03 pounds, after two years 1.03^2, and so on. A logarithmic transformation brings this "geometric" progression onto a linear scale.

2. Plots produced by

```
> logp <- log(p)
> plot.ts(p)
> plot.ts(logp)
```

   Visual features: strong upward trend in both, except it drops sharply and bounces back in two time periods: in the middle and towards the end (namely, financial crisis and the pandemic). Because of the trends, the series do not really appear stationary.

3. `> u = diff(logp)`

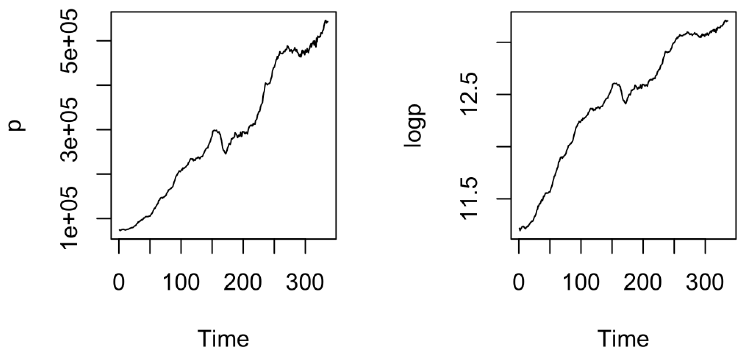

Figure 1: House prices on the natural (left) and log (right) scale.

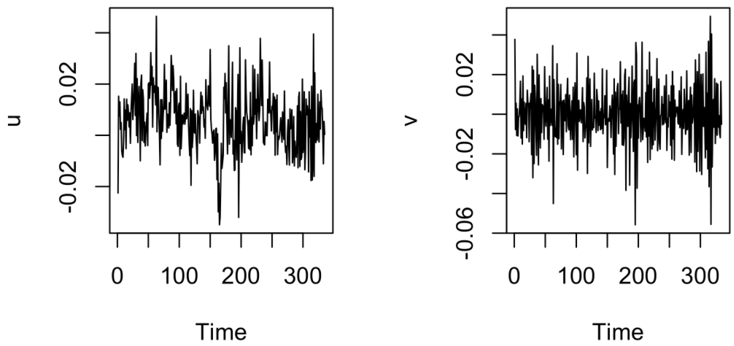

Figure 2: U_t (left) and V_t (right).

```
> v = diff(u)
> plot.ts(u)
> plot.ts(v)
```

See Figure 2. Both series look more stationary than the original one.

4.
```
> acf(u)
> pacf(u)
> acf(v)
> pacf(v)
```

See Figure 3.

5. The ACF and PACF plots suggest that we cannot model U_t or V_t using just AR or MA. Now try:

```
> library(forecast)
> auto.arima(logp,ic="aic")
> auto.arima(logp,ic="bic")
```

They all pointing to a ARIMA(1,2,1) model for $\log P_t$, i.e. ARMA(1,1) for V_t (with zero-mean).

```
Series: logp
ARIMA(1,2,1)

Coefficients:
         ar1      ma1
     -0.1542  -0.7487
s.e.  0.0806   0.0677
```

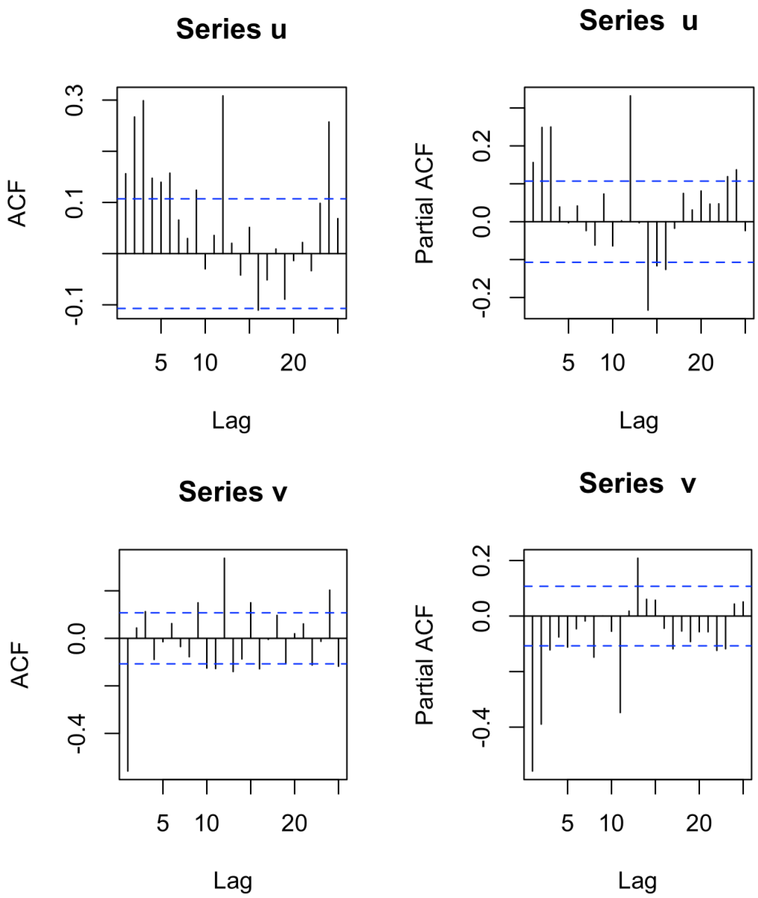

Figure 3: ACF and PACF for U_t (top row) and V_t (bottom row).

```
sigma^2 = 0.00014:  log likelihood = 1009.43
AIC=-2012.86   AICc=-2012.79   BIC=-2001.43
```

However, note that the parameter estimates for ϕ_1 is not significant at 5% (which is fine for the purpose of prediction...)

6.
```
> model<-Arima(logp,c(1,2,1))
> forecast(model,3)
```

We get

```
Point Forecast     Lo 80    Hi 80    Lo 95    Hi 95
337        13.20797 13.19281 13.22313 13.18478 13.23116
338        13.21049 13.18799 13.23300 13.17607 13.24491
339        13.21308 13.18282 13.24333 13.16681 13.25935
```

Then take exponential to converge these figures to the price level.

Part II: Theory

1. (a) Rewriting the process as:

$$(1-0.5B)X_t = (1-1.4B+0.45B^2)\epsilon_t = (1-0.5B)(1-0.9B)\epsilon_t.$$

Since $z=2$ is the common root in AR and MA polynomials, the process can be simplified to

$$X_t = (1-0.9B)\epsilon_t = \epsilon_t - 0.9\epsilon_{t-1}.$$

Therefore, this is actually an MA(1) process, i.e. ARMA(p,q) with $p=0$ and $q=1$.

(b) It is causal (as all MA(q) are causal). It is also invertible, because the root for the MA polynomial (i.e. setting $1-0.9z=0$) lies outside the unit circle.

(c) The ACF for MA(1) is $r_0=1$, $\rho_{-1}=\rho_1=-0.9/1.81$ and $\rho_h=0$ for $|h|>1$.

2. (a) Using the law of iterated expectation,

$$\mathbb{E}(X_t)=\mathbb{E}(\sigma_t\epsilon_t)=\mathbb{E}(\mathbb{E}(\sigma_t\epsilon_t|X_{t-1}))=\mathbb{E}(\sigma_t\mathbb{E}(\epsilon_t|X_{t-1}))=\mathbb{E}(\sigma_t\mathbb{E}(\epsilon_t))=0.$$

In addition,

$$\begin{aligned}\mathbb{E}(X_t^2) &= \mathbb{E}(\sigma_t^2\epsilon_t^2)\\ &= \mathbb{E}((\alpha_0+\alpha_1X_{t-1}^2)\epsilon_t^2)\\ &= (\alpha_0+\alpha_1\mathbb{E}(X_{t-1}^2))\mathbb{E}(\epsilon_t^2)\\ &= \alpha_0+\alpha_1\mathbb{E}(X_t^2),\end{aligned}$$

which yields

$$\mathbb{E}(X_t^2)=\frac{\alpha_0}{1-\alpha_1}.$$

Next,

$$\mathbb{E}(X_t^3)=\mathbb{E}(\sigma_t^3\epsilon_t^3)=\mathbb{E}(\mathbb{E}(\sigma_t^3\epsilon_t^3|X_{t-1}))=\mathbb{E}(\sigma_t^3\mathbb{E}(\epsilon_t^3|X_{t-1}))=\mathbb{E}(\sigma_t^3\mathbb{E}(\epsilon_t^3))=0.$$

Finally,

$$\begin{aligned}X_t^4 &= (\alpha_0+\alpha_1X_{t-1}^2)^2\epsilon_t^4\\ &= (\alpha_0^2+2\alpha_0\alpha_1X_{t-1}^2+\alpha_1^2X_{t-1}^4)\epsilon_t^4.\end{aligned}$$

Since $E\epsilon_t^4=1.8$ (unlike the Gaussian case, which equals 3), hence,

$$\mathbb{E}(X_t^4)=1.8\left(\alpha_0^2+2\alpha_0\alpha_1\frac{\alpha_0}{1-\alpha_1}+\alpha_1^2\mathbb{E}(X_t^4)\right),$$

which gives

$$\mathbb{E}(X_t^4)=\frac{1.8\alpha_0^2(1+\alpha_1)}{(1-\alpha_1)(1-1.8\alpha_1^2)}.$$

(b) Let $v_t = \sigma_t^2(\epsilon_t^2 - 1)$.

$$\mathbb{E}(v_t) = \mathbb{E}[\mathbb{E}(v_t|\mathcal{F}_{t-1})] = \mathbb{E}[\sigma_t^2\mathbb{E}(\epsilon_t^2 - 1|\mathcal{F}_{t-1})] = 0.$$

It is easy to show that $\mathbb{E}v_t^2 < \infty$ (since $\mathbb{E}X_t^6 < \infty$) For any $h > 0$,

$$\mathbb{E}(v_t v_{t+h}) = \mathbb{E}[\mathbb{E}(v_t v_{t+h}|\mathcal{F}_{t+h-1})] = \mathbb{E}[v_t\sigma_{t+h}^2\mathbb{E}(\epsilon_{t+h}^2 - 1|\mathcal{F}_{t+h-1})] = 0.$$

Therefore, $\{v_t\}$ is indeed white noise.

(c) First, for the ACVF of $\{X_t\}$, for any $h > 0$, it is easy to see (by conditioning on $\mathcal{F}_{t+h-1}$) that $\mathbb{E}X_tX_{t+h} = 0$. Therefore, $\gamma_0^X = \frac{a_0}{1-a_1}$ and $\gamma_h^X = 0$ for any $h \neq 0$.
Second, for the ACVF of $\{X_t^2\}$, using the AR(1) representation of ARCH(1), $X_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + v_t$. Therefore,

$$\gamma_h^{X^2} = \alpha_1^{|h|}\mathrm{Var}(X_t^2) = \alpha_1^{|h|}\{\mathbb{E}X_t^4 - (\mathbb{E}X_t^2)^2\},$$

where the values for $\mathbb{E}X_t^4$ and $\mathbb{E}X_t^2$ were derived from the previous sub-question.

(d) From the previous results, we conclude that $\{X_t\}$ is white noise, while $\{X_t^2\}$ is not (as $\alpha_1 \neq 0$).