Solids Mechanics, Including Elastic Wave Propagation.

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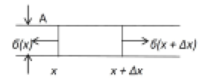
Part 1. Elementary introduction.

1D rod



1. Equilibrium Equation ("Statics")

$$A\{\sigma(x+\Delta x)-\sigma(x)\} = A\rho \int_x^{x+\Delta x} u_{tt}(\xi)d\xi$$

Next, as it follows from the Mean Value theorem

$$\sigma(x+\Delta x)-\sigma(x) = \rho \Delta x u_{tt}(x+\theta\Delta x), \qquad 0 \le \theta \le 1$$

Let us $\Delta x \to 0$

$$\lim_{\Delta x \to 0} \frac{\sigma(x+\Delta x)-\sigma(x)}{\Delta x} = \rho \lim_{\Delta x \to 0} u_{tt}(x+\theta\Delta x)$$

$$\sigma_x = \rho u_{tt}$$

2.Strain ("Geometry")

$$\lim_{\Delta x \to 0} \frac{u(x + \Delta x) - u(x)}{\Delta x} = u_x = \epsilon$$

3. Constitutive relations ("Physics")

A. **Elasticity**

(i) $\sigma = E\epsilon$ - Hook's Law for a Linearly elastic rod
strains are small $\epsilon \ll 1$
Now $\sigma_x = Eu_{xx}$ and $u_{xx} - \frac{1}{c^2}u_{tt} = 0$ with $c = \sqrt{\frac{E}{\rho}}$

(ii) Finite elastic deformations $\epsilon \sim 1$

Say, now $\sigma = E\epsilon + \epsilon^3$ which is an example of physical non-linearity.
Then,

$$\sigma_x = Eu_{xx} + 3\eta\left(\frac{du}{dx}\right)^2 u_{xx}$$

and

$$u_{xx} - \frac{1}{c^2}u_{tt} + \frac{3\eta}{E}u_x^2 u_{xx} = 0$$

If η is small, then it is a room for asymptotics.

B. **Plasticity**

(i) Perfect plasticity

(ii) Plasticity with hardening / softening

C. **Elastic - plastic materials**

(i) Elastic - perfectly plastic material

(ii) Elastic - hardening plastic material

D. **Time - dependent materials**

Small deformations $\sigma = E\epsilon + \mu\epsilon_t$ - Voight material

$$\sigma_x = Eu_{xx} + \mu u_{xxt}$$

$$u_{xx} - \frac{1}{c^2}u_{tt} + \frac{\mu}{E}u_{xxt} = 0$$

Often we get a small μ finalising with a singular perturbed problem.

Part 2. Linear Isotropic Elasticity.

2.1. Stress and Strain tensors and constitutive relations.

1. Stress tensor σ_{ij}

Equilibrium eqns

$$\sigma_{ij,i} = \rho u_{j,tt} \tag{2.1}$$

with $j = 1, 2, 3$ and Einstein convention is assumed

Symmetry : $\sigma_{ij} = \sigma_{ji}$

2. Strain tensor

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,k}) \tag{2.2}$$
$$i, j = 1, 2, 3$$

$$1 \to x|x_1$$
$$2 \to y|x_2$$
$$3 \to z|x_3$$

3. Constitutive relation

$$\sigma_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} \tag{2.3}$$

λ and μ denote Lame constants

Young modulus

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$

Poisson ratio

$$\nu = \frac{\lambda}{2(\lambda + \mu)}$$

usually

$$(0 < \nu < 0.5)$$

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

2.2. Equation of Motion in terms of displacements.

Clear from (2.2) and (2.3)

$$\sigma_{ij,i} = \lambda\delta_{ij}\epsilon_{kk,i} + \mu(u_{i,ji} + u_{j,ii})$$

On substituting into (2.1)

$$(\lambda+\mu)\epsilon_{kk,j} + \mu\Delta u_j = \rho u_{j,tt}$$

Finally,

$$(\lambda+\mu) graddiv \overrightarrow{u} + \mu\Delta\overrightarrow{u} - \rho\overrightarrow{u}_{tt} = 0 \qquad (2.4)$$

where

$$\overrightarrow{u} = (u_1, u_2, u_3)$$

2.3. Shear and Dilatation waves.

$$\overrightarrow{u} = grad\varphi + curl\,\overrightarrow{\psi} \tag{2.5}$$

It follows from (2.5)

$$div\,\overrightarrow{u} = \Delta\varphi$$

On substituting (2.5) into (2.4) and taking into account the last formula

$$grad(c_1^2\Delta\varphi - \varphi_{tt}) + curl(c_2^2\Delta\,\overrightarrow{\psi} - \overrightarrow{\psi_{tt}}) = 0$$

where $c_1^2 = \frac{\lambda+2\mu}{\beta}$, $c_2^2 = \frac{\mu}{\rho}$ denote the speeds of the dilatation and shear waves.

Thus,

$$c_1^2\Delta\varphi - \varphi_{tt} = 0$$

and

$$c_2^2 \Delta \overrightarrow{\psi} - \overrightarrow{\psi_{tt}} = 0 \tag{2.6}$$

4. Rayleigh and Love waves.

4.1. Plane and antiplane strain.

For both of them $\frac{\partial}{\partial x_3} \equiv 0$
Plane strain $u_i = u_i(x_1, x_2), i = 1, 2$ and $u_3 = 0$
Antiplane strain $u_i = 0, u_3 = u_3(x_1, x_2)$

4.2. Rayleigh waves.

Consider plane strain of a half - space $x_2 \geq 0$
Traction free surface

$$x_2 = 0 \quad (\sigma_{22} = \sigma_{21} = 0) \tag{2.7}$$

$$(\sigma_{23} \equiv 0)$$

for plane strain.
Traveling wave solutions

$$\varphi = Ae^{-\alpha x_2 + iq(x_1 - ct)} \tag{2.8}$$

$$\psi_3 = Be^{-\beta x_2 + iq(x_1 - ct)}; \psi_i = 0(i = 1, 2)$$

Conditions : $\alpha > 0, \beta > 0$ correspond to surface waves (decay as $x_2 \to \infty$)
Here c - phase velocity.

On substituting (2.8) into (2.6) and (2.8) into (2.5) we get

$$\alpha = q\sqrt{1 - \frac{c^2}{c_1^2}}, \quad \beta = q\sqrt{1 - \frac{c^2}{c_2^2}}$$

$$u_1 = \varphi_{,1} + \psi_{3,2} = iq(Ae^{-\alpha x_2} - \beta Be^{-\beta x_2})$$

$$u_2 = \varphi_{,2} - \psi_{3,1} = (-\alpha A e^{-\alpha x_2} - iqBe^{-\beta x_2}) \qquad (2.9)$$

Here and below we omit factor $exp[iq(x - ct)]$.
On making use of constitutive relations (2.3) in (2.7) and taking into account geometric relations (2.2) and formulae (2.9) we arrive at homogeneous equations in A and B.
They are

$$\left(2 - \frac{c^2}{c_2^2}\right) A + 2i\sqrt{1 - \frac{c^2}{c_2^2}} B = 0$$

$$-2i\sqrt{1 - \frac{c^2}{c_2^2}} A + \left(2 - \frac{c^2}{c_2^2}\right) B = 0 \qquad (2.10)$$

Solvability of (2.10) yields

$$R(\gamma) = (2 - \gamma^2)^2 - 4\sqrt{(1 - \gamma^2)(1 - \varkappa^2\gamma^2)} = 0$$

With

$$\gamma = \frac{c_r}{c_2}, \quad \varkappa = \frac{c_2}{c_1} = \sqrt{\frac{1-2\gamma}{2-\gamma}} = \sqrt{\frac{\mu}{\lambda+2\mu}} \tag{2.11}$$

R - Rayleigh denominator, $c = c_r$ - Rayleigh wave speed.
We will prove that there exist a root $\gamma < 1$ of $R(\gamma) = 0$ at $0 \leq \nu < \frac{1}{2}$. This root is unique for given ν.

4.3. Love waves.

$$c_2 < c_2^*$$

Antiplane problem

$$\frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} = \frac{1}{c_2^2}\frac{\partial^2 u_3}{\partial t^2}$$

$$\frac{\partial^2 u_3^*}{\partial x_1^2} + \frac{\partial^2 u_3^*}{\partial x_2^2} = \frac{1}{c_2^{*2}}\frac{\partial^2 u_3^*}{\partial t^2} \tag{2.12}$$

Traction free surface $x_2 = -H$, i.e.

$$\sigma_{23} = 0 \Longleftrightarrow \frac{\partial u_3}{\partial x_2} = 0 \tag{2.13}$$

Contact conditions at $x_2 = 0$,

$$u_3 = u_3^*, \sigma_{23} = \sigma_{23}^*$$

or

$$\mu \frac{\partial u_3}{\partial x_2} = \mu_* \frac{\partial u_3^*}{\partial x_2} \tag{2.14}$$

Let us

$$u_3 = f(x_2) e^{iq(x_1 - ct)} \tag{2.15}$$

$$u_3^* = f_*(x_2) e^{iq(x_1 - ct)}$$

On substituting (2.15) into (2.12) we get

$$\frac{\partial^2 f}{\partial x_2^2} + q^2\alpha^2 f = 0, \quad \left(\alpha = \sqrt{\frac{c^2}{c_2^2} - 1}\right)$$

$$\frac{\partial^2 f^*}{\partial x_2^2} - q^2\beta^2 f_* = 0, \quad \left(\beta = \sqrt{1 - \frac{c^2}{c_2^{*2}}}\right) \tag{2.15}$$

Thus, we have a decaing at $x_2 \to \infty$ wave (Love wave):

$$f(x_2) = A\sin(\alpha q x_2) + B\cos(\alpha q x_2)$$

$$f_*(x_2) = Ce^{-\beta q x_2} \tag{2.16}$$

It follows from contact conditions (2.14)

$$B = C, \quad A = -\frac{\mu_*\beta}{\mu\alpha}C$$

Then, the substitution of (2.16) into boundary condition (2.13) at free surface yields

$$\tan(\alpha q H) = \frac{\mu_* \beta}{\mu \alpha} \qquad (2.17)$$

which is the dispersion relation for Love waves. It determines the phase speed versus wave number, i.e. $c = c(qH)$. There are infinitely many Love waves.

Part 3. Lamb (Rayleigh - Lamb)waves.

Consider an infinite layer of thickness $2h$ with fraction free faces

Recall the equations of motion in plane strain

$$\frac{E}{2(1+\nu})\Delta u + \frac{E}{2(1+\nu)*(1-2\nu)} graddivu - \rho\frac{\partial^2 u}{\partial t^2} = 0, \qquad (3.1)$$

where $u = (u_1, u_2, 0)$ is the displacement vector whose components do not depend on $x_3 (u_k = u_k(x_1, x_2, t), k = 1, 2); \Delta$ is Laplacian.

The "displacements - stresses" formulae are

$$\sigma_{11} = \frac{E}{2(1+\nu)\varkappa^2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\nu}{1-\nu}\frac{\partial u_2}{\partial x_2}\right),$$

$$\sigma_{33} = \frac{E\nu}{2(1-\nu^2)\varkappa^2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right), \qquad (3.2)$$

$$\sigma_{22} = \frac{E}{2(1+\nu)\varkappa^2}\left(\frac{\nu}{1-\nu}\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right),$$

$$\sigma_{21} = \frac{E}{2(1+\nu)\varkappa^2}(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}),$$

$$\sigma_{13} = \sigma_{23} = 0.$$

We impose homogeneous boundary conditions on faces $x_2 = \pm h$

$$\sigma_{21} = \sigma_{22} = 0 \tag{3.3}$$

We specify displacement as before in (2.9)

$$u_1 = \frac{\partial \varphi}{\partial x_1} + \frac{\partial \psi_3}{\partial x_2}, u_2 = \frac{\partial \varphi}{\partial x_2} - \frac{\partial \psi_3}{\partial x_1} \tag{3.4}$$

where ϕ and ψ_3 are potentials. Substituting (3.4) into (3.1) we obtain two equations

$$\Delta_2 \varphi - \frac{1}{c_1^2}\frac{\partial^2 \varphi}{\partial t^2}, \quad \Delta_2 \psi_3 - \frac{1}{c_2^2}\frac{\partial^2 \psi_3}{\partial t^2} = 0, \tag{3.5}$$

where

$$\Delta_2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

Let us introduce the dimensionless coordinates $\xi_1 = \frac{x_1}{h}, \zeta = \frac{x_2}{h}$ and $\tau = \frac{tc_2}{h}$ and seek the solution to equations (3.5) in the form

$$\varphi = f(\zeta)\exp[i(K\xi_1 - \Omega\tau)],$$

$$\psi_3 = g(\zeta)\exp[i(K\xi_1 - \Omega\tau)] \tag{3.6}$$

Inserting the latter into (3.5) we have

$$\frac{\partial^2 f}{\partial \zeta^2} - \alpha^2 f = 0, \tag{3.7}$$

$$\frac{\partial^2 g}{\partial \zeta^2} - \beta^2 g = 0, \tag{3.8}$$

where

$$\alpha^2 = K^2 - \varkappa^2\Omega^2, \beta^2 = K^2 - \Omega^2.$$

The vibration modes corresponding to the above equations are separated into two groups. The modes of the first group are symmetric with respect to the midplane of the layer $\zeta = 0$ and those of the second group are antisymmetric. First, examine the **symmetric modes**. For them the displacement u_1 and the stresses $\sigma_{11}, \sigma_{22}, \sigma_{33}$ are even with respect to the thickness variable ζ and the displacement u_2 and the stress σ_{21} are odd. The solutions to the equations (3.7), (3.8) are given by

$$f = A\cosh(\alpha\zeta), \quad g = B\sinh(\beta\zeta) \tag{3.9}$$

where A and B are arbitrary constants. Because of the symmetry it is sufficient to obey the boundary conditions only on one of the faces. The boundary conditions on the other face are satisfied automatically. Expressing the stresses entering into the boundary conditions (3.3) in terms of the functions f and g defined by formulae (3.9) we obtain a system of two linear equations:

$$AKi\alpha\sinh\alpha + B\gamma^2\sinh\beta = 0$$

$$A\gamma^2 \cosh\alpha - BKi\beta\cosh\beta = 0 \tag{3.10}$$

where

$$\gamma^2 = K^2 - \frac{1}{2}\Omega^2.$$

Equating the determinant of this system to zero we obtain the Rayleigh-Lamb dispersion equation [classical works by Lord Rayleigh (1889) and Lamb (1889)]

$$\gamma^4 \cosh\alpha \frac{\sinh\beta}{\beta} - \alpha^2 K^2 \frac{\sinh\alpha}{\alpha}\cosh\beta = 0. \tag{3.11}$$

Displacements and stresses are expressed as

$$u_1 = RKi\left(\gamma^2 \sinh\beta\cosh(\alpha\zeta) - \alpha\beta\sinh\alpha\cosh(\beta\zeta)\right),$$

$$u_2 = R\alpha\left(\gamma^2 \sinh\beta\sinh(\alpha\zeta) - K^2\sinh\alpha\sinh(\beta\zeta)\right),$$

$$\sigma_{11} = \frac{E}{1+\nu)}\frac{R}{h}$$

$$\left(-\gamma^2(K^2 + \frac{\nu}{2(1-\nu)}\Omega^2)\sinh\beta\cosh(\alpha\zeta)) + K^2\alpha\beta\sinh\alpha\cosh(\beta\zeta)\right),$$

$$\sigma_{22} = \frac{E}{1+\nu}\frac{R}{h}\left(\gamma^4\sinh\beta\cosh(\alpha\zeta) - K^2\alpha\beta\sinh\alpha\cosh(\beta\zeta)\right),$$

$$\sigma_{33} = -E\frac{R}{h}\frac{\nu}{2(1-\nu^2)}\gamma^2\Omega^2\sinh\beta\cosh(\alpha\zeta), \qquad (3.12)$$

$$\sigma_{21} = \frac{E}{1+\nu)}\frac{R}{h}iK\alpha\gamma^2\left(\sinh\beta\sinh(\alpha\zeta) - sinh\alpha\sinh(\beta\zeta)\right),$$

In these formulae the factor $\exp[i(K\xi_1 - \Omega\tau)]$ is omitted and R is arbitrary constant.

In case of antisymmetric modes the displacement u_1 and the stresses $\sigma_{11}, \sigma_{22}, \sigma_{33}$ are odd with respect to ζ while the displacement u_2 and the stress σ_{21} are even. The solutions to equations (3.7), (3.8) can be written as

$$f = A\sinh(\alpha\zeta), \quad g = \cosh(\beta\zeta), \tag{3.13}$$

and the Rayleigh-Lamb dispersion equation is

$$\gamma^4 \frac{\sinh\alpha}{\alpha}\cosh\beta - \beta^2 K^2 \cosh\alpha \frac{\sinh\beta}{\beta} = 0. \tag{3.14}$$

The corresponding expressions for stresses and displacements can be obtained from (3.12) by substituting

$\sinh \to \cosh$ and $\cosh \to \sinh.$

We deduce now the asymptotic approximation of the Rayleigh-Lamb equations. First, we consider the *long-wave low-frequency* *approximations* $K << 1, \Omega << 1$. In this case the argument of the hyperbolic functions in (3.11), (3.14) are small. Expanding these functions in Taylor's series we obtain:

for equation (3.11)

$$K^2 = \frac{1-\nu}{2}\Omega^2[1 + O(\Omega^2)] \tag{3.15}$$

for equations (3.14)

$$K^4 = \frac{3(1-\nu)}{2}\Omega^2[1 + O(\Omega)] \tag{3.16}$$

For *symmetric* modes

$$K \sim \Omega \tag{3.17}$$

for the *antisymmetric* modes

$$K \sim \sqrt{\Omega} \tag{3.18}$$

Therefore O-term in asymptotics (3.15), (3.16) is equal to $O(K^2)$.

Consider now *long-wave high-frequency approximations* $(K << 1, \Omega \sim 1)$. It follows directly from the dispersion equations (3.11) and (3.14) that $\cosh\alpha \sinh\beta \sim K^2$, respectively.

Thus, the condition

$$\Omega - \Lambda \sim K^2$$

holds. Here
$\Lambda = \Lambda_{st}$ or $\Lambda = \Lambda_{sh}$
where

$$\Lambda_{st} = \frac{\pi m}{2\varkappa}, \Lambda_{sh} = \frac{\pi m}{2}\,(m = 1, 2, 3, ...) \qquad (3.19)$$

The frequencies Λ_{st} and Λ_{sh} are the co-called **thickness stretch** and **thickness shear resonance frequencies**, respectively. They represent the natural frequencies of an infinitely thin transverse fibre of the layer.

The frequencies Λ_{st} are eigenvalues of the problem

$$\frac{\partial^2 u_2}{\partial \zeta^2} + \varkappa^2 \Omega^2 u_2 = 0 \tag{3.20}$$

with

$$\frac{\partial u_2}{\partial \zeta} = 0 \text{ at } \zeta = \pm 1,$$

which is obtained from problems (3.1)-(3.3) by setting $u_1 = \frac{\partial u_1}{\partial x_1} = 0$. The frequencies Λ_{sh} are eigenvalues of the problem

$$\frac{\partial^2 u_1}{\partial \zeta^2} + \Omega^2 u_1 = 0 \tag{3.21}$$

with

$\frac{\partial u_1}{\partial \zeta} = 0$ at $\zeta = \pm 1$,

which follows from the original plane problem at $u_2 = \frac{\partial u_1}{\partial x_1} = 0$.

The thickness resonance frequencies

$\Lambda_{st} = \Lambda_{st}^s$ and $\Lambda_{sh} = \Lambda_{sh}^s$,

where

$$\Lambda_{st}^s = \frac{\pi(2n-1)}{2\varkappa}, \quad \Lambda_{sh}^s = \pi \quad n\,(n = 1, 2, 3, ...)\,, \tag{3.22}$$

correspond to the dispersion equation (3.11). In their vicinities the following estimates hold

$$K^2 = T_s^{-1}[\Omega^2 - (\Lambda_{st}^s)^2](1 + O(K^2)) \tag{3.23}$$

with

$$T_s = \frac{1}{\varkappa^2} + \frac{8 \cot \Lambda_{st}^2}{\Lambda_{st}^s},$$

and

$$K^2 = P_s^{-1}[\Omega^2 - (\Lambda_{sh}^s)^2] + (1 + O(K^2)) \tag{3.24}$$

with

$$P_s = 1 - \frac{8\varkappa \tan(n\Lambda_{sh}^s)}{\Lambda_{sh}^s}.$$

The thickness resonance frequencies
$\Lambda_{st} = \Lambda_{st}^\alpha$ and $\Lambda_{sh} = \Lambda_{sh}^\alpha$ where

$$\Lambda_{st}^\alpha = \frac{\pi\varkappa}{\varkappa}, \Lambda_{sh}^\alpha = \frac{\pi(2n-1)}{2} \left(n = 1, 2, 3, ...\right), \tag{3.25}$$

correspond to the dispersion equation (3.14). In their vicinities

$$K^2 = T_\alpha^{-1}[\Omega^2 - (\Lambda_{st}^\alpha)^2] + (1 + O(K^2)) \tag{3.26}$$

with

$$T_\alpha = \frac{1}{\varkappa^2} - \frac{8 \tan \Lambda_{st}^\alpha}{\Lambda_{st}^\alpha},$$

and

$$K^2 = P_\alpha^{-1}[\Omega^2 - (\Lambda_{st}^s)^2] + (1 + O(K^2)) \tag{3.27}$$

with

$$P\alpha = 1 + \frac{8\varkappa \cot(\varkappa \Lambda_{st}^\alpha)}{\Lambda_{st}^\alpha}.$$

At $K \sim \Omega \sim 1$ the dispersion equations (3.11),(3.14) do not contain small parameters. In this case simpler *short-wave high-frequency approximations* cannot be constructed. The *short-wave low-frequency approximations* $(K \sim 1, \Omega \ll 1)$ of the Rayleigh-Lamb equations, corresponding to quasi-statics, can be obtained by discarding in equations (3.11), (3.14) terms of order $O(\Omega^2)$ with respect to unity. The result is

$$2K + \sinh(2K) = 0 \tag{3.28}$$

$$2K - \sinh(2K) = 0 \tag{3.29}$$

It is well known that for all non-zero roots of these equations the condition $\operatorname{Im} K \gtrsim 1$ holds. These roots correspond to boundary layers localized in narrow vicinities (of the order of the thickness) of the edges of the layer.

Now we obtain asymptotic formulae for displacements and stresses. At $K \sim \Omega \sim 1$ all the stresses and displacements are of the same asymptotic order, i.e.

$$u_1 \sim u_2, \quad \sigma_{11} \sim \sigma_{22} \sim \sigma_{33} \sim \sigma_{21} \tag{3.30}$$

This conclusion cannot be extended to the long-wave approximations for which the small parameter K^2 enters into formulae (3.12). The leading-approximations of the displacements and stresses are given by:

the low-frequency approximation of the symmetric modes

$K \sim \Omega \ll 1$

$$\begin{aligned}
u_1 &= -R\frac{\nu}{2(1-\nu)}iK\beta\Omega^2, \\
u_2 &= -R\frac{1}{2}\alpha^2\beta\Omega^2\zeta, \\
\sigma_{11} &= E\frac{R}{h}\frac{\nu}{4(1-\nu^2)}\beta\Omega^4, \\
\sigma_{22} &= E\frac{R}{h}\frac{1}{8(1+\nu)}\alpha^2\beta\Omega^4(\zeta^2-1), \\
\sigma_{33} &= -E\frac{R}{h}\frac{\nu}{2(1-\nu^2)}\beta\gamma^2\Omega^2, \\
\sigma_{21} &= E\frac{R}{h}\frac{1}{12(1-\nu^2)}iK\alpha^2\beta\gamma^2\Omega^2(\zeta^3-\zeta),
\end{aligned} \tag{3.31}$$

where

$$\alpha \sim \beta \sim \gamma \sim K$$

the low frequency approximation of the antisymmetric modes $K \sim \sqrt{\Omega} \ll 1$

$$
\begin{aligned}
u_1 &= R\frac{1}{2}iK^2\beta\Omega^2\zeta, \\
u_2 &= -R\frac{1}{2}K\Omega^2, \\
\sigma_{11} &= -E\frac{R}{h}\frac{1}{2(1-\nu^2)}K^3\Omega^2\zeta, \\
\sigma_{22} &= E\frac{R}{h}\frac{1}{12(1-\nu^2)}K^5\Omega^2(\zeta^3-\zeta), \\
\sigma_{33} &= -E\frac{R}{h}\frac{\nu}{2(1-\nu^2)}K^3\Omega^2\zeta, \\
\sigma_{21} &= E\frac{R}{h}\frac{1}{4(1-\nu^2)}iK^4\Omega^2(\zeta^2-1),
\end{aligned}
\tag{3.32}
$$

the high-frequency approximation of the symmetric modes in the vicinities of the thickness stretch resonance frequencies

$(K \ll 1, \Omega \sim 1, \Omega - \Lambda^s_{st} \sim K^2)$

$$u_1 = RK(\Lambda^s_{st})^2 \left(\frac{1}{2} \sin \Lambda^s_{st} \cos(\Lambda^s_{st} \varkappa \zeta) + (-1)^n \varkappa \cos(\Lambda^s_{st} \zeta) \right),$$

$$u_2 = Ri\frac{\varkappa}{2}(\Lambda^s_{st})^3 \sin \Lambda^s_{st} \sin(\varkappa \Lambda^s_{st} \zeta), \qquad (3.33)$$

$$\sigma_{11} = E\frac{R}{h}i\frac{\nu}{4(1-\nu^2)}(\Lambda^s_{st})^4 \sin \Lambda^s_{st} \cos(\varkappa \Lambda^s_{st} \zeta),$$

$$\sigma_{22} = E\frac{R}{h}\frac{1}{4(1+\nu)}\Lambda^s_{st})^4 sin \Lambda^s_{st} \cos(\varkappa \Lambda^s_{st} \zeta),$$

$$\sigma_{33} = E\frac{R}{h}\frac{\nu}{4(1-\nu^2)}(\Lambda^s_{st})^4 \sin \Lambda^s_{st} \cos(\varkappa \Lambda^s_{st} \zeta),$$

$$\sigma_{21} = E\frac{R}{h}\frac{\varkappa}{2(1+\nu)}K(\Lambda^s_{st})^3 \left(- \sin \Lambda^s_{st} \sin(\varkappa \Lambda^s_{st} \zeta) + (-1)^{n+1} \sin(\Lambda^s_{st} \zeta)\right)$$

the high-frequency approximation of the symmetric modes in the vicinities of the thickness shear resonance frequencies $(K << 1, \Omega \sim 1, \Omega - \Lambda^s_{st} \sim K^2)$

$$u_1 = -R\varkappa K(\Lambda^s_{st})^2 \sin(\varkappa\Lambda^s_{st}) \cos(\Lambda^s_{st}\zeta),$$

$$u_2 = Ri\varkappa K^2 \Lambda^s_{st} \tan(\varkappa\Lambda^s_{st})$$
$$\left((-1)^{n+1} 2\varkappa \sin(\varkappa\Lambda^s_{st}\zeta) + \cos(\varkappa\Lambda^s_{st}) \sin(\Lambda^s_{st}\zeta)\right),$$

$$\sigma_{11} = \frac{ER}{h} i \frac{\varkappa}{1+\nu} K^2 (\Lambda^s_{sh})^2 \tan(\varkappa\Lambda^s_{sh})$$
$$\left((-1)^{n+1} \frac{\nu}{1-\nu} \cos(\varkappa\Lambda^s_{sh}\zeta) - \cos(\varkappa\Lambda^s_{sh}) \cos(\Lambda^s_{sh}\zeta)\right),$$

$$\sigma_{22} = \frac{ER}{h} i \frac{\varkappa}{1+\nu} K^2 (\Lambda^s_{sh})^2 \tan(\varkappa \Lambda^s_{sh})$$

$$\left((-1)^{n+1} \cos(\varkappa \Lambda^s_{sh} \zeta) - \cos(\varkappa \Lambda^s_{sh}) \cos(\Lambda^s_{sh} \zeta)\right),$$

$$\sigma_{33} = \frac{ER}{h} i (-1)^{n+1} \frac{\varkappa \nu}{1-\nu^2} K^2 (\Lambda^s_{sh})^2 \tan(\varkappa \Lambda^s_{sh}) \cos(\varkappa \Lambda^s_{sh} \zeta),$$

$$\sigma_{21} = \frac{ER}{h} \frac{\varkappa}{2(1+\nu)} K (\Lambda^s_{sh})^3 \sin(\varkappa \Lambda^s_{sh}) \sin(\Lambda^s_{sh} \zeta);$$

the hight-frequency approximation of the antisymmetric modes in the vicinities of the thickness stretch resonance frequencies $(K << 1, \Omega \sim 1, \Omega - \Lambda^s_{sh} \sim K^2)$

$$u_1 = RK(\Lambda^a_{sh})^2 \left(\tfrac{1}{2}\cos\Lambda^a_{sh}\sin(\varkappa\Lambda^a_{sh}\zeta) + (-1)^{n+1}\varkappa\sin(\Lambda^a_{sh}\zeta)\right),$$

$$u_2 = -Ri\frac{1}{2}\varkappa(\Lambda^a_{sh})^3\cos(\Lambda^a_{sh})\cos(\varkappa\Lambda^a_{sh}\zeta),$$

$$\sigma_{11} = \frac{ER}{h}i\frac{\nu}{4(1-\nu^2)}(\Lambda^a_{sh})^4\cos(\Lambda^a_{sh})\sin(\varkappa\Lambda^a_{sh}\zeta), \quad (3.35)$$

$$\sigma_{22} = \frac{ER}{h}i\frac{1}{4(1+\nu)}(\Lambda^a_{sh})^4\cos(\Lambda^a_{sh})\sin(\varkappa\Lambda^a_{sh}\zeta),$$

$$\sigma_{33} = \frac{ER}{h}i\frac{\nu}{4(1-\nu^2)}(\Lambda^a_{sh})^4\cos(\Lambda^a_{sh})\sin(\varkappa\Lambda^a_{sh}\zeta),$$

$$\sigma_{21} = \frac{ER}{h}\frac{\varkappa}{2(1+\nu)}K(\Lambda^a_{sh})^3$$

$$\left(\cos(\Lambda^a_{sh})\cos(\varkappa\Lambda^a_{sh}\zeta) + (-1)^{n+1}\cos(\Lambda^a_{sh}\zeta)ight\right);$$

the high-frequency approximation of the antisymmetric modes in the vicinities of the thickness shear resonance frequencies $(K \ll 1, a = 1, \Omega \sim 1, \Omega - \Lambda_{sh}^{s} \sim K^2)$

$$u_1 = -R\varkappa K(\Lambda_{sh}^{a})^2 \cos(\varkappa\Lambda_{sh}^{a}) \sin(\Lambda_{sh}^{a}\zeta), \tag{3.36}$$

$$u_2 = Ri\varkappa K^2\Lambda_{sh}^{a} \cot(\varkappa\Lambda_{sh}^{a}) \Big((-1)^{(n+1)} 2\varkappa \cos(\varkappa\Lambda_{sh}^{a}\zeta) - \sin(\varkappa\Lambda_{sh}^{a}) \cos(\Lambda_{sh}^{a}\zeta)\Big)$$

$$\sigma_{11} = \frac{ER}{h} i \frac{\nu}{1+\nu} K^2 (\Lambda_{sh}^{a})^2 \cot(\varkappa\Lambda_{sh}^{a}) \left((-1)^n \frac{\nu}{1-\nu} \sin(\varkappa\Lambda_{sh}^{a}\zeta) - \sin(\varkappa\Lambda_{sh}^{a}) \sin(\Lambda_{sh}^{a}\zeta)\right),$$

$$\sigma_{22} = \frac{ER}{h} i \frac{\varkappa}{1+\nu} K^2 (\Lambda^a_{sh})^2 \cot(\varkappa \Lambda^a_{sh})$$
$$\left((-1)^n \sin(\varkappa \Lambda^a_{sh} \zeta) + \sin(\varkappa \Lambda^a_{sh}) \sin(\Lambda^a_{sh} \zeta)\right),$$

$$\sigma_{33} = \frac{ER}{h} i (-1)^n \frac{\varkappa \nu}{1-\nu^2} K^2 (\Lambda^a_{sh})^2 \cot(\varkappa \Lambda^a_{sh}) \sin(\varkappa \Lambda^a_{sh} \zeta),$$

$$\sigma_{21} = -\frac{ER}{h} \frac{\varkappa}{2(1+\nu)} K (\Lambda^a_{sh})^3 \cos(\varkappa \Lambda^a_{sh}) \cos(\Lambda^a_{sh} \zeta);$$

The asymptotic error of formulae (3.31) - (3.36) coincides with that of the leading approximations of the roots of the Rayleigh-Lamb dispersion equations and is equal to $O(K^2)$.

Examining formulae (3.31) - (3.36) we arrive at the following relations:

in the case of formulae (3.31)

$$u_2 \sim K u_1, \quad \sigma_{21} \sim K\sigma_{22} \sim K^3 \sigma_{ii} \tag{3.37}$$

$$\sigma_{ii} \sim \frac{E}{h} u_1 (i = 1, 3)$$

in the case of formulae (3.32)

$$u_1 \sim K u_2, \quad \sigma_{22} \sim K\sigma_{21} \sim K^2 \sigma_{ii} \tag{3.38}$$

$$\sigma_{ii} \sim \frac{E}{h} K^2 u_2 \quad (i = 1, 3)$$

in the case of formulae (3.33) and (3.35)

$$u_1 \sim K u_2, \sigma_{31} \sim K\sigma_{kk} \tag{3.39}$$

$$\sigma_{kk} \sim \frac{E}{h} u_2 \quad (i = 1, 2, 3)$$

in the case of formulae (3.34) and (3.36)

$$u_2 \sim K u_1, \quad \sigma_{kk} \sim K \sigma_{21} \tag{3.40}$$

$$\sigma_{21} \sim \frac{E}{h} u_1 \quad (i = 1, 2, 3)$$

The follows from the asymptotics above that the low-frequency approximation of the symmetric modes and the high-frequency approximation in the vicinities of the thickness shear resonance frequencies are tangential $(u_1 >> u_2)$ while the low-frequency approximation of the antisymmetric modes and the high-frequency approximation in the vicinities of the thickness stretch resonance frequencies are transverse $(u_1 \ll u_2)$.

Part 4. Elastic Plate Bending.

The model problem considered in the previous section provide important preliminary information for asymptotic derivation of various approximate equations. We now use the latter to deduce the 1D equations of plate bending from the 2D equations of elasticity corresponding to plane strain of a layer.

Let us determine the asymptotic behavior of the stresses and strains in the layer by the formulae

$$u_1 = R\eta u_1^*, \quad u_2 = R u_2^*, \quad \sigma_{ii} = E\sigma_{ii}^* \tag{4.1}$$
$$\sigma_{21} = E\eta^2\sigma_{21}^*, \quad \sigma_{22} = E\eta^3\sigma_{22}^*, \quad (i = 1, 3)$$

Where $\frac{h}{R} << 1$ - small geometric parameter R - a typical wave-length.

Here all the quantities with the asterisk are of the same asymptotic order.

The asymptotics proposed coincide with asymptotics (3.38) corresponding to the long-wave low-frequency approximation of the antisymmetric modes of a layer in the case of plane strain. In back, $K \sim \eta << 1$ and $\Omega \sim \eta$. Such a choice is in agreement with the usual idea. In this case, plate bending represents a long-wave low-frequency state which is antisymmetric with respect to the midplane.

We also assume that $\xi_1 = R^{-1}x_1, \varsigma = \frac{x_2}{h}, t = \eta^{-2}(\frac{c_2}{R})^{-1}\tau$, i.e. in the same manner as for the long-wave low-frequency approximation of the antisymmetric modes (see (3.18)).

We substitute formulae (4..1) into equations of part 1. Then, we write down the equations obtained in an easy-to-use form

$$\frac{\partial u_2^*}{\partial \zeta} = \eta^4 \sigma_{22}^* - \eta^2 \nu(\sigma_{11}^* + \sigma_{22}^*),$$

$$\frac{\partial u_1^*}{\partial \zeta} = -\frac{\partial u_2^*}{\partial \xi_1} + \eta^2 2(1+\nu)\sigma_{21}^*,$$

$$\sigma_{11}^* = \frac{1}{1-\nu^2}\frac{\partial u_1^*}{\partial \xi_1} + \eta^2 \frac{\nu}{1-\nu}\sigma_{22}^*, \tag{4.2}$$

$$\frac{\partial \sigma_{21}^*}{\partial \zeta} = -\frac{\partial \sigma_{11}^*}{\partial \xi_1} + \eta^2 \frac{1}{2(1+\nu)}\frac{\partial^2 u_1^*}{\partial \tau^2},$$

$$\frac{\partial \sigma_{22}^*}{\partial \zeta} = -\frac{\partial \sigma_{21}^*}{\partial \xi_1} + \frac{1}{2(1+\nu)}\frac{\partial^2 u_2^*}{\partial \tau^2}.$$

We also assume that differentiation with respect to the variables ξ_1 and τ does not change the asymptotic order of unknown quantities.

The boundary conditions on faces take from

$$\sigma_{21}^* = \sigma_{22}^* = 0 \quad at \quad \zeta = \pm 1. \tag{4.3}$$

Formulae (4.2) show that to within the error $O(\eta^2)$

(i) the variation of the length of the normal element,

(ii) the transverse shear deformation,

(iii) the Poisson influence of the stress component σ_{22} on σ_{11},

(iv) the tangential forces of inertia
are asymptotically negligible (see $(4.2) - (4.2)$, respectively). The factors enumerated define the **Kirchhoff hypotheses** in the theory of plate bending. The analogous hypotheses in the shell theory are known as the Kirchoff-Love hypotheses.

Let us construct the leading asymptotic approximation of the problem (4.2)-(4.3). First, we neglect terms of order $O(\eta^2)$ in equations (4.2).

They become

$$
\begin{aligned}
\frac{\partial u_2^*}{\partial \zeta} &= 0, \\
\frac{\partial u_1^*}{\partial \zeta} &= -\frac{\partial u_2^*}{\partial \xi_1}, \\
\sigma_{11}^* &= \frac{1}{1-\nu^2}\frac{\partial u_1^*}{\partial \xi 1}, \\
\sigma_{22}^* &= \frac{\nu}{1-\nu^2}\frac{\partial u_1^*}{\partial \xi_1}, \\
\frac{\partial \sigma_{21}^*}{\partial \zeta} &= -\frac{\partial \sigma_{11}^*}{\partial \xi_1}, \\
\frac{\partial \sigma_{22}^*}{\partial \zeta} &= -\frac{\partial \sigma_{21}^*}{\partial \xi_1} + \frac{1}{2(1+\nu)}\frac{\partial^2 u_2^*}{\partial \tau^2}.
\end{aligned}
\tag{4.4}
$$

Integrating the latter equations with respect to ζ we establish the dependence of the unknown quantities on the transverse coordinate

$$
\begin{aligned}
u_2^* &= u_2^{(0)}, \\
u_1^* &= \zeta u_1^{(1)}, \\
\sigma_{11}^* &= \zeta \sigma_{11}^{(1)}, \\
\sigma_{22}^* &= \zeta \sigma_{33}^{(1)}, \\
\sigma_{21}^* &= \sigma_{21}^{(0)} + \zeta^2 \sigma_{21}^{(2)}, \\
\sigma_{22}^* &= \zeta \sigma_{22}^{(1)} + \zeta^3 \sigma_{22}^{(3)}.
\end{aligned}
\tag{4.5}
$$

All the quantities with a suffix in parentheses do not depend on the variable ζ and are related by the formulae (the boundary conditions (4.3) are taken into account)

$$
u_1^{(1)} = -\frac{\partial u_2^{(0)}}{\partial \xi_1},
$$

$$
\begin{aligned}
\sigma_{11}^{(1)} &= \frac{1}{1-\nu^2}\frac{\partial u_1^{(1)}}{\partial \xi_1}, \\
\sigma_{22}^{(1)} &= \frac{\nu}{1-\nu^2}\frac{\partial u_1^{(1)}}{\partial \xi_1}, \\
\sigma_{21}^{(2)} &= -\frac{1}{2}\frac{\partial \sigma_{11}^{(1)}}{\partial \xi_1}, \\
\sigma_{21}^{(0)} &= -\sigma_{21}^{(2)}, \\
\sigma_{22}^{(1)} &= -\frac{\partial \sigma_{21}^{(0)}}{\partial \xi_1} + \frac{1}{2(1+\nu)}\frac{\partial^2 u_2^{(0)}}{\partial \tau^2}, \\
\sigma_{22}^{(3)} &= -\frac{1}{3}\frac{\partial \sigma_{21}^{(2)}}{\partial \xi_1}, \\
\sigma_{22}^{(1)} &+ \sigma_{22}^{(3)} = 0.
\end{aligned}
\tag{4.6}
$$

Relations (4.6) represent a system of eight equations in eight unknown 1D quantities. The system does not contain a small parameter. It follows from this that all the quantities with a suffix in parentheses and, consequently, all the quantities with the asterisk are of the same asymptotic order. This reasoning justifies asymptotics (4.1).

Formulae (4.6) define the basic relations of the approximate theory of plate bending in the 1D case. The governing equation of plate bending in terms of the transverse displacement of the midplane can be easily derived from them. Let us express all the quantities entering into (4.6) in terms of component $u_2^{(0)}$

$$u_1^{(1)} = -\frac{\partial u_2^{(0)}}{\partial \xi_1},$$

$$\sigma_{11}^{(1)} = -\frac{1}{1-\nu^2}\frac{\partial^2 u_2^{(0)}}{\partial \xi_1^2},$$

$$\sigma_{22}^{(1)} = \frac{\nu}{1-\nu^2}\frac{\partial^2 u_2^{(0)}}{\partial \xi_1^2},$$

$$\sigma_{21}^{(2)} = \frac{1}{2(1-\nu^2)}\frac{\partial^3 u_2^{(0)}}{\partial \xi_1^3},$$

$$\sigma_{21}^{(0)} = -\frac{1}{2(1-\nu^2)}\frac{\partial^3 u_2^{(0)}}{\partial \xi_1^3}, \tag{4.7}$$

$$\sigma_{22}^{(1)} = \frac{1}{2(1-\nu^2)}\frac{\partial^4 u_2^{(0)}}{\partial \zeta_1^4} + \frac{1}{2(1+\nu)}\frac{\partial^2 u_2^{(0)}}{\partial \tau^2},$$

$$\sigma_{22}^{(3)} = -\frac{1}{6(1-\nu^2)}\frac{\partial^4 u_2^{(0)}}{\partial \xi_1^4}.$$

Substituting now the expressions for the components $\sigma_{22}^{(1)}$ and $\sigma_{22}^{(3)}$ into the formula (4.6) we obtain the following 1D equation

$$\frac{1}{3(1-\nu^2)}\frac{\partial^4 u_2^{(0)}}{\partial \zeta_1^4} + \frac{1}{2(1+\nu)}\frac{\partial^2 u_2^{(0)}}{\partial \tau^2} = 0, \tag{4.8}$$

which coincides with the classical equation of plate bending in the 1D case.

Let us $w = Ru_2^{(0)}$. Then the Kirchhoff equation becomes

$$D\frac{\partial^4 w}{\partial x_1^4} + 2\rho h\frac{\partial^2 w}{\partial t^2} = 0. \tag{4.9}$$

With $D = \frac{2Eh^3}{3(1-\nu^2)}$