Measure Theory   Fifth Week

Integration

With $(X, \mathcal{A})$ a measurable space,

$\mathcal{S}$ is the collection of simple functions and

$\mathcal{S}_+$ is the collection of non-negative simple functions.

χ_A is the function such that $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$.

If μ is also a measure defined on $\mathcal{A}$,

and $f = \sum_{i=1}^{n} a_i \chi_{A_i} \quad \forall i \; a_i \in \mathbf{R}$

for finitely many disjoint $A_1, \ldots, A_n \in \mathcal{A}$

define $\int f d\mu = \sum_{i=1}^{n} a_i \mu(A_i)$

(where $0 \cdot \infty = \infty \cdot 0 = 0$).

Need to know that $\int f \, d\mu$ is well defined:

Suppose $g = f$ and $g = \sum_{j=1}^{k} b_j \chi_{B_j}$:

We can break down both g and f further as simple functions by the disjoint sets

$(A_i \cap B_j \mid i = 1, \ldots, n \quad j = 1, \ldots, k)$

(assuming $X = \cup_i A_i = \cup_j B_j$)

and $f = \sum_i \sum_j a_i \chi_{A_i \cap B_j}$ and

$g = \sum_i \sum_j b_j \chi_{A_i \cap B_j}$.

But where $A_i \cap B_j \neq \emptyset$ by $f = g$ it must be that $a_i = b_j$

and where $A_i \cap B_j = \emptyset$ it doesn't matter,

because $\mu(A_i \cap B_j) = 0$.

Therefore $\int g \, d\mu$ is equal to $\sum_i \sum_j a_i \mu(A_i \cap B_j)$,

and by $\sum_j \mu(A_i \cap B_j) = \mu(A_i)$

we have that $\int g \, d\mu = \int f \, d\mu$.

The simple functions defined on a measurable space $(X, \mathcal{A})$ form a vector subspace:

if f is a simple function then αf is also a simple function for any $\alpha \in \mathbf{R}$,

if f, g are simple functions then $f + g$ is a simple function.

The latter is true by taking the collection

$$(A_i \cap B_j \mid i = 1, \ldots, n \quad j = 1, \ldots, k)$$

where the $A_1, \ldots, A_n$ define f and the $B_1, \ldots, B_k$ define g.

The natural question is whether integration is a linear functional on the subspace of simple functions.

Lemma:

$\int \alpha f \, d\mu = \alpha \int f \, d\mu$ and

$\int (f + g) \, d\mu = \int f \, d\mu + \int g \, d\mu$.

Proof:

Let $A_1, \ldots, A_n$ and $a_1, \ldots, a_n$ define f.

αf is defined by the same sets and $a'_i = \alpha a_i$, therefore $\int \alpha f \; d\mu = \sum_i \alpha a_i \mu(A_i) = \alpha(\sum_i a_i \mu(A_i)) = \alpha \int f \; d\mu$.

Let $B_1, \ldots, B_k$ and $b_1, \ldots, b_k$ define g.

$f + g$ is defined by $a_i + b_j$ and the $(A_i \cap B_j \mid i = 1, \ldots, n \quad j = 1, \ldots, k)$:

$\int (f+g) \; d\mu = \sum_i \sum_j (a_i + b_j) \mu(A_i \cap B_j) = \sum_i \sum_j a_i \mu(A_i \cap B_j) + \sum_i \sum_j b_j \mu(A_i \cap B_j) = \int f \; d\mu + \int g \; d\mu$.

Lemma: If $f \leq g$ for simple functions f, g then $\int f \; d\mu \leq \int g \; d\mu$.

Proof: $g = f + (g - f)$

and $g - f$ is a simple function in $\mathcal{S}_+$.

Lemma: Let $f \in \mathcal{S}_+$

and let $f_1 \leq f_2 \leq \ldots$ be a sequence of simple functions in $\mathcal{S}_+$

such that for each x

$f(x) = \lim_{i\to\infty} f_i(x)$.

Then $\int f \; d\mu = \lim_{i\to\infty} \int f_i \; d\mu$.

As $f_i \leq f$ for every i,

it follows that $\int f_i \, d\mu \leq \int f \, d\mu$.

For any $\epsilon > 0$ with ϵ strictly less than any positive value of f,

define simple functions g_i

by $g_i(x) = \min(f_i(x), f(x) - \epsilon)$.

Define $B_i := \{x \mid g_i(x) < f(x) - \epsilon\}$:

p.w. convergence $\Rightarrow \cap_{i=1}^{\infty} B_i = \emptyset$.

There are two cases, $\int f = \infty$ and $\int f < \infty$.

If $\int f = \infty$,

there must be some r with $r > \epsilon$ and a set A with $f|_A = r$ and $\mu(A) = \infty$:

Suppose there was a bound $M < \infty$ such that $\mu(A \backslash B_i) \leq M$ for all i.

The sets $A_i := A \backslash B_i$ would be a non-decreasing sequence and $\tilde{A} := \cup_i A_i$ would be a subset of measure no more than M,

meaning that $A \backslash \tilde{A}$ would be a set of infinite measure where $\lim_{i \to \infty} f_i(x) \leq f(x) - \epsilon$, a contradiction.

Therefore for every $M > 0$ it follows that $\lim_{i \to \infty} \int g_i d\mu \geq M(r - \epsilon)$,

and hence $\lim_{i \to \infty} \int g_i d\mu = \infty$ and the same for f_i.

In the other case, $\int f < \infty$, define $A := \{x \mid f(x) > 0\}$ with $M = \mu(A)$.

Because f is simple with $\int f < \infty$, it follows that $M < \infty$ and $\forall i\ \mu(B_i) < M$.

By the continuity of meausure

$\lim_{i\to\infty} \mu(B_i) = 0.$

Because simple functions have finite values, f has a maximum positive value $N < \infty$

and it follows from $\lim_{i\to\infty} \mu(B_i) = 0$ that

$\lim_{i\to\infty} \int g_i\, d\mu \geq -\epsilon M + \int f\, d\mu.$

The rest follows by $g_i \leq f_i$ for every i and the arbitrary choice of ϵ. $\square$

Let f be a measurable function $f : X \to [0, \infty]$.

The integral $\int f \, d\mu$ is defined to be

$\sup_{g \in \mathcal{S}_+,\ g \leq f} \int g \, d\mu$.

Lemma: Let $f : X \to [0, \infty]$ be a measurable function

and let $f_1 \leq f_2 \leq \ldots$ be a sequence of simple functions in $\mathcal{S}_+$

such that for each x

$f(x) = \lim_{i \to \infty} f_i(x)$.

Then $\int f \, d\mu = \lim_{i \to \infty} \int f_i \, d\mu$.

Proof: Assume first that $\int f d\mu < \infty$. For any given $\epsilon > 0$ let g be a simple function such that $g \leq f$ and

$\int g \; d\mu \geq -\epsilon + \int f \; d\mu,$

(by definition of the integral it exists).

As the $\tilde{f}_i = f_i \wedge g$ are also simple functions

with $\lim_{i \to \infty} \tilde{f}_i(x) = g(x)$ for all x,

it follows that

$\lim_{i \to \infty} \int \tilde{f}_i \; d\mu = \int g \; d\mu \geq -\epsilon + \int f \; d\mu.$

The rest follows from $\tilde{f}_i \leq f_i \Rightarrow$

$\lim_{i \to \infty} \int \tilde{f}_i \; d\mu \leq \lim_{\to \infty} \int f_i \; d\mu.$

And if $\int f d\mu = \infty$ do the same with any $M > 0$ and $0 \leq g \leq f$ with $\int g \; d\mu \geq M$.

Monotone Convergence Theorem:

Let $f : X \to [0, \infty]$ and $f_i : X \to [0, \infty]$ be measurable functions

such that $f_1 \le f_2 \le \dots$

such that for each x

$f(x) = \lim_{i \to \infty} f_i(x)$.

Then $\int f \, d\mu = \lim_{i \to \infty} \int f_i \, d\mu$.

Proof: By previous lemma, there is a sequence $(g_l \mid l = 1, 2, \dots)$ of simple functions

with $g_l \leq f$ for every l. and

$\lim_{l\to\infty} g_l(x) = f(x)$ for every x.

By the last lemma $\lim_{l\to\infty} \int g_l \, d\mu = \int f \, d\mu$.

For every $i = 1, 2, \dots$ there are simple function $h^i_j \in \mathcal{S}_+$

with $h^i_1 \leq h^i_2, \dots$ and $\lim_{j\to\infty} h^i_j(x) = f_i(x)$

and $\lim_{j\to\infty} \int h^i_j \, d\mu = \int f_i \, d\mu$.

For every $l = 1, 2, \dots$

define $f^l_k = \vee_{i,j \leq k}(h^i_j \wedge g_l)$.

We have $f^l_1 \leq f^l_2 \leq \dots$ and $\forall i \quad f^l_i \leq f_i$.

Choosing any x and $\epsilon > 0$ there is an i such that $f_i(x) \geq f(x) - \frac{\epsilon}{2}$ and then there is a j such that $h^i_j(x) \geq f_i(x) - \frac{\epsilon}{2}$.

This means that $\lim_{j\to\infty} f^l_j(x) = g_l(x)$

and so $\lim_{j\to\infty} \int f^l_j \, d\mu = \int g_l \, d\mu$.

And with $f^l_j \leq f_j$ for all j it follows that

$\lim_{j\to\infty} \int f_j \, d\mu \geq \int g_l \, d\mu$.

But with $\lim_{j\to\infty} \int f_j \, d\mu \leq \int f d\mu$

and $\lim_{l\to\infty} \int g_l \, d\mu = \int f \, d\mu$,

$\Rightarrow \lim_{j\to\infty} \int f_j \, d\mu = \int f \, d\mu$. $\square$

Note: The same concluson holds for the more liberal condition $\lim_{i\to\infty} f_i(x) = f(x)$ for almost all x,

since one can restict all arguments to the set where the equality holds and the complement of this set contributes nothing to the integrals.

Any measurable $f : X \to [-\infty, +\infty]$

is called *integrable* if

both $\int f^+ \, d\mu$ and $\int f^- \, d\mu$ are finite.

If either $\int f^+ \, d\mu$ or $\int f^- \, d\mu$ is finite, then $\int f \, d\mu$ is defined to be

$$\int f^+ \, d\mu \quad - \quad \int f^+ \, d\mu$$

If A is a measurable set and f a measurable function

then $\int_A f \, d\mu = \int \chi_A f \, d\mu$, given that it is well defined.

Fatou's Lemma:

Let $f_1, f_2, \dots$ be a sequence of non-negative valued measurable functions.

Then $\int \liminf_n f_n \, d\mu \leq \liminf_n \int f_n \, d\mu$.

Proof: Let $g_n = \inf_{k=n}^{\infty} f_k$.

We have $g_1 \leq g_2 \leq \cdots \leq g_n \leq f_n$ and

$\lim_{n\to\infty} g_n(x) = \liminf_n f_n(x)$ for all x.

By the monotone convergence theorem,

$\int \liminf_n f_n \, d\mu = \int \lim_n g_n \, d\mu = \lim_n \int g_n \, d\mu =$

$\liminf_n \int g_n \, d\mu \leq \liminf_n \int f_n \, d\mu$.

Dominated Convergence Theorem

Let $g : X \to [0, \infty)$ be an integrable function and

let f and $f_1, f_2, \ldots$ be $[-\infty, +\infty]$ valued measurable functions

such that $f(x) = \lim_n f_n(x)$ almost everywhere

and $|f_n(x)| \leq g(x)$.

Then $\int f \, d\mu = \lim_n \int f_n \, d\mu$.

Proof:

By Fatou's Lemma

$\int \liminf_i (g + f_i)\, d\mu \le \liminf_i \int (g + f_i)\, d\mu$,

$\int \liminf_i (g - f_i)\, d\mu \le \liminf_i \int (g - f_i)\, d\mu$.

As $\int g\, d\mu$ is finite, we can cancel it out and

$\int \liminf_i f_i\, d\mu \le \liminf_i \int f_i\, d\mu$

and $\int \limsup_i f_i\, d\mu \ge \limsup_i \int f_i\, d\mu$.

As $\limsup_i f_i = \liminf_i f_i$

and $\limsup_i$ of a sequence is never smaller than $\liminf_i$ of the same sequence,

all four values must be equal.