Theory of Linear Models
Exercises 2

21 January 2019

1. For the multiple regression model given by $\boldsymbol{\beta}' = [\beta_0\ \beta_1\ \beta_2]$ and

$$\mathbf{X} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}:$$

   (a) find rank($\mathbf{X}$);

   (b) find a generalized inverse of $\mathbf{X}'\mathbf{X}$;

   (c) hence find a least squares estimator of $\boldsymbol{\beta}$;

   (d) check whether or not β_1 is estimable;

   (e) check whether or not $\beta_1 + \beta_2$ is estimable.

2. Consider the half-replicate fractional factorial design for three factors, each at two levels, in four runs:

X_1	X_2	X_3
-1	-1	-1
-1	1	1
1	-1	1
1	1	-1

Consider fitting the model

$$\mu = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 + \beta_{23} x_2 x_3,$$

with the usual covariance assumptions.

(a) Write down $\mathbf{X}$.

(b) Find rank($\mathbf{X}$).

(c) Find a generalized inverse of $\mathbf{X}'\mathbf{X}$.

(d) Hence find a least squares estimator of $\boldsymbol{\beta}$.

(e) Check whether or not β_1 is estimable.

(f) Check whether or not $\beta_1 - \beta_{23}$ is estimable.

(g) Obtain the variance of the least squares estimator of $\beta_1 - \beta_{23}$.

3. Derive the maximum likelihood estimator of σ^2 in the general linear model.

4. Write down a linear model for which the maximum likelihood estimator of σ^2 is the minimum mean square error estimator.

5. Explain what would be done by an M-estimator of $\boldsymbol{\beta}$ with

$$\rho(u) = \begin{cases} u^2 & a \leq u \leq a; \\ 0 & \text{otherwise.} \end{cases}$$

(Actually, this is not quite well-defined. We need some side conditions, such as having a such that at least n_0 observations have $-a \leq \epsilon_i/s \leq a$. You don't need to think about this to answer the question.)

6. Consider fitting a multiple regression model to the following data:

Y	X_1	X_2	X_3
10	-1	-1	-1.0001
12	-1	1	-0.9999
20	1	-1	1.0000
22	1	1	1.0000

(a) Calculate $(\mathbf{X}'\mathbf{X} + \lambda\mathbf{I})^{-1}$ for $\lambda = 0, 10^{-8}, 10^{-7}, \ldots, 1$. (I suggest using R or another package which inverts matrices.)

(b) Obtain the ridge regression estimates of β_1, β_2 and β_3 in each of these cases.

(c) Comment on the results.