LTCC Geometry and Physics: Mock exam answers

(**Note:** *The Einstein summation convention is assumed in question 3.)*

1. As a subset of the $n \times n$ complex matrices $\mathrm{Mat}_n(\mathbb{C})$, the $n \times n$ Hermitian matrices are

$$\mathcal{H}_n := \{ X \in \mathrm{Mat}_n(\mathbb{C}) \mid X = X^\dagger \},$$

where the dagger $\dagger$ denotes the Hermitian conjugate (complex conjugate transpose).
(i) By writing down an explicit set of coordinates, show that, as a real manifold, $\mathcal{H}_n$ is isomorphic to $\mathbb{R}^d$ for some dimension d that you should find in terms of n.
(ii) Letting $\dot{X}$ denote the tangent to a path $X(t) \in \mathcal{H}_n$, a metric g on the space of $n \times n$ Hermitian matrices is defined in terms of the trace by

$$g(\dot{X}, \dot{X}) = \mathrm{tr}\, \dot{X}^2.$$

By considering the action $S = \int_{t_0}^{t_1} L\, \mathrm{d}t$ with the Lagrangian

$$L = \tfrac{1}{2}\, g(\dot{X}, \dot{X}) = \tfrac{1}{2}\, \mathrm{tr}\, \dot{X}^2$$

and calculating the Euler-Lagrange equations in terms of the coordinates from part (i), or otherwise, show that the geodesics for this metric are straight lines.

Solution to Q1: (i) We can write the entries of $X \in \mathcal{H}_n$ as

$$X_{jj} = x_j, \qquad X_{jk} = \overline{X}_{kj} = y_{jk} + \mathrm{i} z_{jk}, \quad j < k,$$

for real coordinates x_j, $j = 1, \ldots, n$ and y_{jk}, z_{jk}, $1 \le j < k \le n$, which gives a single real chart with $d = n + 2 \times \frac{1}{2} n(n-1) = n^2$ real coordinates, so the $n \times n$ Hermitian matrices are isomorphic to $\mathbb{R}^{n^2}$ as a manifold.
(ii) As in lectures, for a Riemannian manifold with metric g, the geodesic equations are the Euler-Lagrange equations derived from the action $S = \int L\, \mathrm{d}t$ with Lagrangian $L = \frac{1}{2} g(\dot{x}, \dot{x})$, where $\dot{x}$ denotes the tangent vector to a path parametrized by t in a set of local coordinates (x). Using the coordinates from part (i), the Lagrangian takes the explicit form

$$L = \frac{1}{2} \sum_{j,k=1}^{n} \dot{X}_{jk} \dot{X}_{kj} = \frac{1}{2} \sum_j \dot{X}_{jj}^2 + \sum_{j<k} \dot{X}_{jk} \dot{\overline{X}}_{jk},$$

which is just

$$L = \frac{1}{2} \sum_j \dot{x}_j^2 + \sum_{j<k} (\dot{y}_{jk}^2 + \dot{z}_{jk}^2)$$

(corresponding to free motion). Hence the Euler-Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{x}_j} \right) - \frac{\partial L}{\partial x_j} = \ddot{x}_j = 0, \qquad j = 1, \ldots, n,$$

and similarly for the coordinates y_{jk}, z_{jk} we have the equations

$$2\ddot{y}_{jk} = 0 = 2\ddot{z}_{jk}, \qquad j < k.$$

The solutions of all these equations are linear functions of t, which give straight lines in $\mathcal{H}_n$, as required. **Otherwise:** Replacing $X \to X + \delta X$ in the action $S[X]$ gives

$$S[X + \delta X] = \frac{1}{2} \int_{t_0}^{t_1} \mathrm{tr}\big(\dot{X}^2 + \dot{X}\,\delta\dot{X} + \delta\dot{X}\,\dot{X} + (\delta\dot{X})^2)\big)\,\mathrm{d}t,$$

so combining the two middle terms and integrating by parts gives

$$S[X + \delta X] = S[X] + \int_{t_0}^{t_1} \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{tr}(\dot{X}\,\delta X) - \mathrm{tr}(\ddot{X}\,\delta X) + \tfrac{1}{2}(\delta\dot{X})^2 \right)\,\mathrm{d}t.$$

Assuming that the variation δX vanishes at the endpoints t_0, t_1, but is otherwise arbitrary, the first term in the middle above is $[\mathrm{tr}(\dot{X}\,\delta X)]_{t_0}^{t_1} = 0$, so from the principle of least action, requiring that the first variation $\delta S = 0$ gives the equation of motion

$$\ddot{X} = 0 \implies X = At + B, \qquad A, B \quad \text{arbitrary},$$

which is straight line motion. (In lectures, it was mentioned that geodesics can also be derived from variations of the arc length integral $\int \mathrm{d}s = \int \sqrt{g(\dot{x}, \dot{x})}\,\mathrm{d}t$, which gives yet another way to obtain the same result.)

2. The n-particle Calogero-Moser system on $T^*\mathbb{R}^n$, with coordinates/momenta q_j, p_j, $j = 1, \ldots, n$ and canonical symplectic structure $\omega = \sum_{j=1}^n \mathrm{d}p_j \wedge \mathrm{d}q_j$, is defined by the Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^n p_j^2 + \sum_{1 \le j < k \le n} \frac{1}{(q_j - q_k)^2}.$$

(i) Write down the equations of motion (Hamilton's equations).
(ii) For two particles ($n = 2$), show that these equations imply that the Lax equation

$$\frac{\mathrm{d}L}{\mathrm{d}t} = [M, L]$$

holds, where $L = L(t), M = M(t)$ are 2×2 matrices with entries given by

$$L_{jj} = p_j, \qquad L_{jk} = \frac{\mathrm{i}}{q_j - q_k} \quad (j \neq k), \qquad M_{jj} = 0, \qquad M_{jk} = \frac{\mathrm{i}}{(q_j - q_k)^2} \quad (j \neq k),$$

with $\mathrm{i} = \sqrt{-1}$. Hence show that are two independent conserved quantities

$$H_j = \frac{1}{j}\,\mathrm{tr}\,L^j, \qquad j = 1, 2,$$

and use this to conclude that the two-particle system is integrable in the Liouville sense.
(iii) Letting $Q = Q(t) = \mathrm{diag}(q_1, q_2)$, find a constant matrix C such that the (Hermitian) matrix

$$X(t) = Q_0 + L_0 t \quad \text{with} \quad Q_0 = Q(0),\ L_0 = L(0),$$

satisfies the momentum map condition

$$[X, \dot{X}] = C = [Q, L].$$

Solution to Q2: **(i)** Hamilton's equations are

$$\dot{q}_j = p_j, \qquad \dot{p}_j = 2\sum_{k\neq j}\frac{1}{(q_j-q_k)^3}, \qquad j=1,\dots,n.$$

(ii) Computing the Lax equation, the left-hand side gives

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \begin{pmatrix} \dot{p}_1 & -\frac{\mathrm{i}(\dot{q}_1-\dot{q}_2)}{(q_1-q_2)^2} \\ \frac{\mathrm{i}(\dot{q}_1-\dot{q}_2)}{(q_1-q_2)^2} & \dot{p}_2 \end{pmatrix},$$

and the right-hand side is

$$[M,L] = \left[\begin{pmatrix} 0 & \frac{\mathrm{i}}{(q_1-q_2)^2} \\ \frac{\mathrm{i}}{(q_1-q_2)^2} & 0 \end{pmatrix}, \begin{pmatrix} p_1 & \frac{\mathrm{i}}{(q_1-q_2)} \\ -\frac{\mathrm{i}}{(q_1-q_2)} & p_2 \end{pmatrix}\right],$$

so the Lax equation follows from $\dot{p}_1 = -\dot{p}_2 = 2(q_1-q_2)^{-3}$, $\dot{q}_1-\dot{q}_2 = p_1-p_2$ (only 3 independent conditions). Taking the trace of the Lax equation gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{tr}\,L = \mathrm{tr}\,[M,L] = 0, \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\,\tfrac{1}{2}\mathrm{tr}\,L^2 = \mathrm{tr}\,L\dot{L} = \mathrm{tr}\,(L[M,L]) = 0$$

(as in lectures), so this produces two conserved quantities

$$H_1 = \mathrm{tr}\,L = p_1+p_2, \qquad H_2 = \tfrac{1}{2}\mathrm{tr}\,L^2 = H = \tfrac{1}{2}(p_1^2+p_2^2) + \frac{1}{(q_1-q_2)^2},$$

and from the dependence on momenta these are clearly independent functions. Also, because H_1 is conserved, the Poisson bracket of these two functions is given by $\frac{\mathrm{d}}{\mathrm{d}t}H_1 = \{H_1,H_2\} = 0$, so they are in involution. The system has 2 degrees of freedom and has 2 independent conserved quantities in involution, so it satisfies the Liouville definition of complete integrability.
(iii) Direct calculation shows that $[X,\dot{X}] = [Q_0,L_0] = [Q,L] = C$, where

$$C = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

(This suggests a connection between Calogero-Moser and free motion as in question 1, but that's another story! The practical upshot of this is that the system can be solved by diagonalizing the Hermitian matrix X to find Q.)
3. In 1+1-dimensional Minkowski spacetime with coordinates $(x^0,x^1)=(t,x)$ and metric $g=(g_{\mu\nu})=\mathrm{diag}(1,-1)$, a φ^4 field theory is defined by the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}g^{\mu\nu}\varphi_\mu\varphi_\nu - \tfrac{1}{4}\lambda\big(1-\varphi^2\big)^2.$$

Here $g^{-1}=(g^{\mu\nu})$ is the co-metric, subscripts on φ denote derivatives, $\lambda>0$ is a coupling constant, and units are chosen so that the speed of light $c=1$.
(i) Write down the Euler-Lagrange equations for this theory, and use the momentum density

$$\pi = \frac{\partial\mathcal{L}}{\partial\varphi_t}$$

to obtain an expression for the Hamiltonian via the standard Legendre transformation

$$H = \int_{\mathbb{R}} \Big(\pi \varphi_t - \mathcal{L} \Big) \, \mathrm{d}x.$$

(ii) Consider a stationary field ($\varphi_t = 0$) that interpolates between the two different vacua at $\varphi = \pm 1$ as $x \to \pm\infty$, and complete the square in the integrand to show that the value of energy $H = E =$const can be written as

$$E = \frac{1}{2} \int_{-\infty}^{\infty} \Big(\varphi_x - \sqrt{\tfrac{\lambda}{2}} (1 - \varphi^2) \Big)^2 \, \mathrm{d}x + \sqrt{\tfrac{\lambda}{2}} \int_{-\infty}^{\infty} (1 - \varphi^2) \varphi_x \, \mathrm{d}x.$$

Hence, by rewriting the second term as an integral over φ, obtain the Bogomolny-Prasad-Sommerfield (BPS) bound

$$E \geq \frac{2\sqrt{2\lambda}}{3},$$

and sketch the profile of a topological soliton (a kink) which attains this bound.
(**Note:** Obtaining the explicit solution of the differential equation is not necessary to answer the question.)

Solution to Q3: (i) From

$$\mathcal{L} = \tfrac{1}{2}(\varphi_t^2 - \varphi_x^2) - \tfrac{\lambda}{4}\big(1 - \varphi^2\big)^2,$$

we have the Euler-Lagrange equations

$$\frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial \varphi_\mu} \right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0 \implies \varphi_{tt} - \varphi_{xx} - \lambda \varphi (1 - \varphi^2) = 0.$$

The momentum density is

$$\pi = \frac{\partial \mathcal{L}}{\partial \varphi_t} = \varphi_t,$$

and the Hamiltonian has the standard form $\int_{\mathbb{R}^d} \big(\tfrac{1}{2}(\pi^2 + (\nabla\varphi)^2) + \mathcal{V}(\varphi) \big) \, \mathrm{d}^d x$ considered in lectures, being an integral over d-dimensional space in the case $d = 1$, namely

$$H = \int_{\mathbb{R}} \Big(\tfrac{1}{2}\pi^2 + \tfrac{1}{2}\varphi_x^2 + \tfrac{\lambda}{4}(1 - \varphi^2)^2 \Big) \, \mathrm{d}x.$$

(ii) Setting $\varphi_t = \pi = 0$ gives the value of the energy $H = E$ for a stationary solution as

$$E = \frac{1}{2} \int_{\mathbb{R}} I(x) \, \mathrm{d}x,$$

where the integrand is

$$I(x) = \varphi_x^2 + \frac{\lambda}{2}\big(1 - \varphi^2\big)^2 = \left(\varphi_x - \sqrt{\frac{\lambda}{2}}(1 - \varphi^2) \right)^2 + 2\sqrt{\frac{\lambda}{2}}(1 - \varphi^2)\varphi_x,$$

which gives the required result for the energy. Since the first term is a perfect square, this gives the BPS bound

$$E \geq \tfrac{1}{2} \int_{-\infty}^{\infty} 2\sqrt{\tfrac{\lambda}{2}} (1 - \varphi^2) \varphi_x \, \mathrm{d}x = \sqrt{\tfrac{\lambda}{2}} \int_{-1}^{1} (1 - \varphi^2) \, \mathrm{d}\varphi,$$

using the given boundary conditions for φ as $x \to \pm\infty$, or in other words

$$E \geq \sqrt{\tfrac{\lambda}{2}}\Big[\varphi - \tfrac{1}{3}\varphi^3\Big]_{-1}^{1} = \frac{2\sqrt{2\lambda}}{3},$$

as required. The minimum energy bound is saturated when the squared term in $I(x)$ vanishes, which reduces the second order ODE for $\varphi(x)$ to first order, that is

$$\varphi_x = \sqrt{\tfrac{\lambda}{2}}(1-\varphi^2) \geq 0,$$

which is consistent with the boundary conditions. The explicit kink solution is

$$\varphi = \tanh\left(\sqrt{\tfrac{\lambda}{2}}(x-a)\right),$$

where the constant a is arbitrary, but the shape of the kink profile (as sketched below) can be inferred simply from the asymptotic values $\varphi = \pm 1$ and the fact that $\varphi_x > 0$ for $|\varphi| < 1$.

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Sketch: