

Theory of Linear Models

Exam Question - Solution

Mock 2026

April 2025

The following data on a response variable Y were collected from an experiment with explanatory variables X_1 , X_2 and X_3 .

X_1	X_2	X_3	Y
-1	-1	-1	12.3
-1	-1	1	19.7
-1	1	-1	14.3
-1	1	1	13.9
1	-1	-1	4.6
1	-1	1	5.7
1	1	-1	2.2
1	1	1	8.1
0	0	0	19.4
0	0	0	1.5
0	0	0	1.1
0	0	0	19.7

The proposed model is the second order response surface model (including a constant) with Y_i 's uncorrelated and $V(Y_i) = \sigma^2$.

1. Express the model in matrix form, i.e. write down the model matrix $\mathbf{X}$ and the vector of parameters $\boldsymbol{\beta}$.

$$E(Y) = X\beta \text{ and } V(Y) = \sigma^2 I, \text{ where}$$

$$X = \begin{bmatrix} 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{and } \beta' = [\beta_0 \ \beta_1 \ \beta_2 \ \beta_3 \ \beta_{11} \ \beta_{22} \ \beta_{33} \ \beta_{12} \ \beta_{13} \ \beta_{23}].$$

2. Find the rank of the matrix $\mathbf{X}$.

It is immediately obvious that columns 1-5 and 8-10 are linearly independent, but the 6th and 7th are equal to the 5th. Hence $\text{rank}(X) = 8$.

3. Find a generalized inverse of $\mathbf{X}'\mathbf{X}$.

$$X'X = \begin{bmatrix} 12 & 0 & 0 & 0 & 8 & 8 & 8 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 8 & 8 & 8 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 8 & 8 & 8 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 8 & 8 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}.$$

Let

$$X^* = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

so that

$$X^{*'}X^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$X'X + X'X = \begin{bmatrix} 12 & 0 & 0 & 0 & 8 & 8 & 8 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 10 & 7 & 7 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 7 & 9 & 8 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 7 & 8 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

and

$$(X'X)^- = (X'X + X'X)^{-1}$$

$$= \begin{bmatrix} 0.25 & 0 & 0 & 0 & -0.08333 & -0.08333 & -0.08333 & 0 & 0 & 0 \\ 0 & 0.125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.125 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.125 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.08333 & 0 & 0 & 0 & 0.2639 & -0.06944 & -0.06944 & 0 & 0 & 0 \\ -0.08333 & 0 & 0 & 0 & -0.06944 & 0.5972 & -0.4028 & 0 & 0 & 0 \\ -0.08333 & 0 & 0 & 0 & -0.06944 & -0.4028 & 0.5972 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.125 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.125 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.125 \end{bmatrix}.$$

4. Find a least squares estimator $\hat{\beta}$ of β . (Hint: Use the generalized inverse of $X'X$.)

$$\begin{aligned}
X'Y &= \begin{bmatrix} Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7 + Y_8 + Y_9 + Y_{10} + Y_{11} + Y_{12} \\ Y_5 + Y_6 + Y_7 + Y_8 - Y_1 - Y_2 - Y_3 - Y_4 \\ Y_3 + Y_4 + Y_7 + Y_8 - Y_1 - Y_2 - Y_5 - Y_6 \\ Y_2 + Y_4 + Y_6 + Y_8 - Y_1 - Y_3 - Y_5 - Y_7 \\ Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7 + Y_8 \\ Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7 + Y_8 \\ Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7 + Y_8 \\ Y_1 + Y_2 + Y_7 + Y_8 - Y_3 - Y_4 - Y_5 - Y_6 \\ Y_1 + Y_3 + Y_6 + Y_8 - Y_2 - Y_4 - Y_5 - Y_7 \\ Y_1 + Y_4 + Y_5 + Y_8 - Y_2 - Y_3 - Y_6 - Y_7 \end{bmatrix} \\
\Rightarrow \hat{\beta} &= \begin{bmatrix} (Y_9 + Y_{10} + Y_{11} + Y_{12})/4 \\ (Y_5 + Y_6 + Y_7 + Y_8 - Y_1 - Y_2 - Y_3 - Y_4)/8 \\ (Y_3 + Y_4 + Y_7 + Y_8 - Y_1 - Y_2 - Y_5 - Y_6)/8 \\ (Y_2 + Y_4 + Y_6 + Y_8 - Y_1 - Y_3 - Y_5 - Y_7)/8 \\ 0.0417(Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7 + Y_8) - 0.0833(Y_9 + Y_{10} + Y_{11} + Y_{12}) \\ 0.0417(Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7 + Y_8) - 0.0833(Y_9 + Y_{10} + Y_{11} + Y_{12}) \\ 0.0417(Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7 + Y_8) - 0.0833(Y_9 + Y_{10} + Y_{11} + Y_{12}) \\ (Y_1 + Y_2 + Y_7 + Y_8 - Y_3 - Y_4 - Y_5 - Y_6)/8 \\ (Y_1 + Y_3 + Y_6 + Y_8 - Y_2 - Y_4 - Y_5 - Y_7)/8 \\ (Y_1 + Y_4 + Y_5 + Y_8 - Y_2 - Y_3 - Y_6 - Y_7)/8 \end{bmatrix}.
\end{aligned}$$

5. Show that β_{33} is not estimable.

$\beta_{11} = c'\beta$, where $c' = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0]$. Hence

$$I - (X'X)^{-}X'X = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and so $c'\{I - (X'X)^{-}X'X\} = [0 \ 0 \ 0 \ 0 \ -\frac{1}{3} \ -\frac{1}{3} \ \frac{2}{3} \ 0 \ 0 \ 0] \neq 0$, so β_{33} is not estimable.

6. Show that $\theta = \beta_{11} + \beta_{22} + \beta_{33}$ is estimable and find the form of its least squares estimator $\hat{\theta}$ and its observed value.

$\theta = c'\beta$, where $c' = [0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0]$. Hence $c'\{I - (X'X)^{-}X'X\} = [0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]$, so θ is estimable. $\hat{\theta} = c'\hat{\beta} = (Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7 + Y_8)/8 - (Y_9 + Y_{10} + Y_{11} + Y_{12})/4$. This has observed value -0.325 .

7. Find the variance of $\hat{\theta}$.

$$V(\hat{\theta}) = \sigma^2 c'(X'X)^{-}c = 3\sigma^2/8.$$

8. Write down the form of a $100(1 - \alpha)\%$ confidence interval for θ .

A $100(1 - \alpha)\%$ CI for θ is

$$\hat{\theta} \pm t_{4;1-\alpha/2} \sqrt{3s^2/8},$$

where s^2 is the residual mean square.