FTSI, 2024

(1) (a) This is just algebra.

$$(\hat{\mu}_i - \mu)^T(\hat{\mu}_i - \mu) = (Y-\mu)^T(Y-\mu) - 2c_i\frac{(Y-\mu)^T Y}{Y^T Y} + \frac{c_i^2}{Y^T Y},$$

so $L(\mu, \hat{\mu}_1) < L(\mu, \hat{\mu}_2)$

$$\equiv \frac{2(c_1-c_2)(Y-\mu)^T Y}{Y^T Y} > \frac{c_1^2 - c_2^2}{Y^T Y}$$

$$\equiv (Y-\mu)^T Y > \frac{1}{2}\frac{c_1^2 - c_2^2}{c_1 - c_2}, \text{ as } c_1 > c_2$$

$$\equiv \left(Y - \frac{\mu}{2}\right)^T\left(Y - \frac{\mu}{2}\right) - \frac{\mu^T\mu}{4} > \frac{1}{2}(c_1 + c_2).$$

Then $PN_\mu(\hat{\mu}_1, \hat{\mu}_2)$ follows as $Y - \frac{\mu}{2} \sim N_p(\frac{\mu}{2}, I)$, so that $PN_\mu(\hat{\mu}_1, \hat{\mu}_2) = P(\chi^2(p, \lambda) > \frac{1}{2}(c_1+c_2)+\lambda)$, with $\lambda = \mu^T\mu/4$.

(b) Let $c > c^*$. By the result in (a), with $c_1 = c$, $c_2 = c^*$, $PN_\mu(\hat{\mu}_c, \hat{\mu}_{c^*}) = P(\chi^2(p,\lambda) > \frac{1}{2}(c+c^*)+\lambda)$.

But, $\frac{1}{2}(c+c^*) > c^*$, so

$PN_\mu(\hat{\mu}_c, \hat{\mu}_{c^*}) < P(\chi^2(p,\lambda) > c^* + \lambda)$

$= P(\chi^2(p,\lambda) > \chi^2_{0.5}(p,\lambda)) = \frac{1}{2}$. So, $\hat{\mu}_c$ is inadmissible. Similarly, if $c < c^*$, we have

$PN_\mu(\hat{\mu}_{c^*}, \hat{\mu}_c) = P(\chi^2(p,\lambda) > \frac{1}{2}(c+c^*) + \lambda)$

$> P(\chi^2(p,\lambda) > c^* + \lambda) = \frac{1}{2}$, so that

$PN_\mu(\hat{\mu}_c, \hat{\mu}_{c^*}) = 1 - PN_\mu(\hat{\mu}_{c^*}, \hat{\mu}_c) < \frac{1}{2}$.

Again, $\hat{\mu}_c$ is inadmissible. So, $\hat{\mu}_c$ is inadmissible if $c \neq c^*$ and the oracle value is c^*, as claimed. $\hat{\mu}_{c^*}$ is not an estimator, as it depends on the unknown μ, through λ.

(c). Using (a) again with $c_1 = p-2$, $c_2 = 0$,

$PN_\mu(\hat{\mu}_{p-2}, Y) = P(\chi^2(p,\lambda) > \frac{p-2}{2} + \lambda)$

$= P(\chi^2(p,\lambda) > \frac{p}{2} - 1 + \lambda) > P(\chi^2(p,\lambda) > p-1+\lambda)$

$\geq \frac{1}{2}$, since Robert gives $\chi^2_{0.5}(p,\lambda) \geq p-1+\lambda$.

So, Y is inadmissible in terms of closeness.

Similarly, take $c_1 = p-1$, $c_2 = p-2$ to obtain

$PN_\mu(\hat{\mu}_{p-1}, \hat{\mu}_{p-2}) = P(\chi^2(p,\lambda) > p - \frac{3}{2} + \lambda)$

$> P(\chi^2(p,\lambda) > p-1+\lambda) \geq \frac{1}{2}$, so taking $c = p-1$

gives an estimator dominating the James-Stein estimator, which is therefore inadmissible.

(d) We know, from lectures, $E_0 D = E_0 L(0, \hat{\mu}_{p-2}) - E_0 L(0, Y) = 2 - p$.

Then, $P_0(D < E_0 D) = P_0(L(0, \hat{\mu}_{p-2}) - L(0, Y) < 2-p)$

$$= P_0\left(Y^T Y - 2(p-2) + \frac{(p-2)^2}{Y^T Y} - Y^T Y < 2-p\right)$$

$$= P_0\left(\frac{(p-2)^2}{Y^T Y} < p-2\right) = P_0(Y^T Y > p-2)$$

$$= P(\chi^2(p,0) > p-2) > P(\chi^2(p,0) > p-1)$$

$\geq \frac{1}{2}$, by Robert's result that $\chi^2_{0.5}(p,0) \geq p-1$.

This ┌do some numerical evaluations, say in R ┘ decreases as a function of p.

(2). (a). The joint pdf is of the form

$$f(y : \mu, \lambda) = h(y) \exp\left\{ \frac{-\lambda}{2\mu^2} \sum y_i - \frac{\lambda}{2} \sum y_i^{-1} + \frac{n\lambda}{\mu} + \frac{n}{2} \log \lambda \right\}, \quad \text{—}*$$

which constitutes a full $(2,2)$ exponential family, with natural parameters $\phi^1 = -\frac{\lambda}{2\mu^2}$, $\phi^2 = -\frac{\lambda}{2}$ and

natural statistics $S_1(Y) = \sum_1^n Y_i$, $S_2(Y) = \sum_1^n \frac{1}{Y_i}$.

(b). If μ is known, we can write

$$f(y; \lambda) = \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \left(\prod_1^n \frac{1}{y_i^{3/2}}\right) \exp\left\{-\frac{\lambda}{2} T(y)\right\},$$

where $T(Y) = \sum_1^n \frac{(Y_i - \mu)^2}{\mu^2 Y_i}$ is the natural statistic,

complete since we now have a full (1,1) exponential family. From the distributional result given and 'Sum of independent chi-squared is chi-squared' we have $\lambda T(Y) \sim \chi^2_n$. Then [seen in lectures],

$E\left(\frac{1}{\lambda T(Y)}\right) = \frac{1}{n-2}$, $E\left(\frac{n-2}{T(Y)}\right) = \lambda$. Since

$\frac{n-2}{T(Y)}$ is unbiased and function of the complete natural statistic [which is minimal sufficient], it is MVUE.

(c). If μ is unknown, from $*$ we see that the required test is equivalent to a test on ϕ^2 with ϕ^1

a nuisance parameter. The theory of optimal testing in full exponential families then implies that an optimal (UMPU) test of size α of H_0 against H_1 is a conditional test, of the form: reject H_0 when $\sum Y_i^{-1} < K$, where $K \equiv K(\bar{Y})$ satisfies

$$P_{\lambda_0}\left(\sum_1^n Y_i^{-1} < K \mid \bar{Y} = \bar{y}\right) = \alpha.$$

We observe that $V = \sum_1^n (Y_i^{-1} - \bar{Y}^{-1})$ is an increasing function of $\sum_1^n Y_i^{-1}$ for fixed $\bar{Y}$ and is, from given result, independent of $\bar{Y}$. Then, by a 'useful result' given in lectures, the UMPU test is equivalent to a test based on the marginal distribution of V. We have $\lambda V \sim \chi^2_{n-1}$, so reject H_0 if $V < c$, where $\lambda_0 c$ is the α quantile of χ^2_{n-1}, for a UMPU test of size α.

(3). To provoke thought. Unbiasedness [point estimator, test, general decision rule] presented as means of restricting (sensibly) class of inference procedures. Not an optimality criterion.